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SEMICENTENNIAL PUBLICATIONS

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UNIVERSITY OF CALIFORNIA

1868-1918

ELECTRICAL PHENOMENA
IN
PARALLEL CONDUCTORS

VOLUME I
ELEMENTS OF TRANSMISSION

BY

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FIRST EDITION

NEW YORK
JOHN WILEY & SONS, Inc.
London: CHAPMAN & HALL, Limited
1918

Stanhope Press
F. H. GILSON COMPANY
BOSTON, U.S.A.

PREFACE

It is of fundamental importance to be able to predetermine with as much numerical accuracy as possible, the phenomena which may take place in electrical systems. It is of importance first, because such predetermination, when possible, can be made at much less expense and trouble than is required to obtain the same information experimentally; and second, because information so obtained opens a way to new developments and improvements in the application of scientific knowledge. To predetermine in a numerical way what may happen in electrical systems under different conditions requires that the various phenomena be expressed in a mathematical form or the equivalent thereof. Of course it is not the mathematical expressions themselves which are of main interest, but these expressions are necessary since they afford the only means at our disposal, other than experimental investigation, by which the desired knowledge may be obtained.

In this volume an attempt has been made to set forth the mathematical developments leading to solutions for a number of problems arising in connection with the transmission of electrical energy over metallic circuits. The fundamental equations for the propagation of currents and voltages along an electrical circuit consisting of parallel conductors or the equivalent thereof have been developed and presented by a number of writers — Heaviside, Fleming, Pupin, Steinmetz, Kennelly and others; but it has been the writer's experience to note that the application of these developments to present-day engineering problems has been comparatively rare. This has been due partly to the unfamiliarity of a great many of engineers with the particular type of mathematics necessarily involved in such discussions, and partly to the fact that the presentation of the material in various publications has not been in such a form as particularly to invite its application. Instead of developing and applying to problems rigorous methods of solution, the tendency has been to develop apparently simple approximate methods; but investigation shows that in many cases

the simplicity is only apparent and that accurate results may be obtained with as little mechanical labor as the approximate. Further, a realization or understanding of accurate methods leads to an ability to develop approximate methods where such are needed on account of the complexity of rigorous solutions, and to apply these approximate methods with due consideration of their limitations and significance.

For some time the author has felt that it would be of real service to have collected under one head as much as possible of the available material dealing with the present subject, as well as the addition of such further developments as may be possible. It seemed desirable, also, to include in this work the discussion of a number of matters, which, though not falling directly under the above title, are nevertheless used so often in connection with the particular problems which do fall under this head that a complete understanding thereof is essential. Accordingly, in this volume a discussion of periodic and alternating quantities in general, with methods of analysis of periodic curves into Fourier's series, has been included. A discussion of the more common form of oscillograph, methods for its use and an analysis of its mode of operation, was deemed desirable. The fact that at the present date the oscillograph plays such an important part in the study of transmission-line phenomena leads to the belief that its discussion here is not irrelevant.

In regard to the treatment of transmission-line phenomena proper, this volume is to be thought of as forming an introduction to subsequent volumes dealing with specialized forms of electrical transmission, and therefore it naturally has for its scope a discussion of some of the more general properties of transmission systems only. Continuously alternating-current phenomena only have been discussed, for the introduction of methods for determining the transient currents or voltages in a conductor or system of conductors leads to solutions more complicated than seem advisable here. Further, desirable forms of such solutions differ so greatly for different types of circuits which may arise that no one particular form could be considered sufficiently general to be treated in this introductory volume.

Between points of discontinuity in a simple line carrying continuously alternating electrical quantities, the currents and voltages at one point are related to the currents and voltages at

another point by simple linear relations, and these relations are naturally expressed in terms of hyperbolic functions. Since in alternating-current work complex quantities are used for the representation, analytically, of the vectors in a vector diagram, our algebraic formulas necessarily involve hyperbolic functions of complex variables. This fact has been one of the greatest obstacles preventing a more general utilization of rigorous transmission-line formulas. The portion of this volume which deals with transmission-line phenomena may properly be thought of as merely an elaboration dealing with the various forms and results which may be obtained from the two fundamental linear equations relating quantities at one point in a line to similar quantities at another point. In many places these linear equations are of exactly the same form as the equations for systems of entirely different types, — that is, localized circuits, — and therefore our discussions, instead of applying to transmission lines only, are applicable also to the general electrical circuit for which equations of the same nature hold true. The coefficients of such linear equations are of course given by different functional relations among the various circuit constants when different types of circuits are considered. In "Theory and Calculation of Electric Currents" by J. L. LaCour and O. J. Bragstad a discussion of the general electrical circuit is given, with reference to the operating characteristics of circuits carrying uniformly alternating electrical quantities, which the author has found very illuminating. It is to be hoped that further treatments from this standpoint may appear, for the generality of the methods employed make such discussions of great value.

Although, in this present volume, some explanation of the complex-quantity method of dealing with alternating quantities seemed desirable in order to establish for the reader a clear knowledge of the forms of procedure followed, it has necessarily been assumed that those interested would be familiar with the various fundamental theorems and ideas concerning the behavior of the simple properties which go to make up an electrical circuit. Having clearly in mind the physical significance of the four fundamental constants involved in electrical systems, that is, resistance, conductance, self-inductance and electrostatic capacity, and having formulated the methods by which mathematical discussions relating to alternating quantities may be carried on, further

developments may be thought of as only mathematical in their nature. The above condition, however, does not obviate the necessity of being very familiar, from a physical standpoint, with the particular problems in hand, for unless such familiarity exists much difficulty will be encountered in carrying through the mathematical developments leading to solutions which are of interest and importance to the practicing engineer and to the physicist.

The author wishes to express here his appreciation of the valuable advice and encouragement offered by Professor Harris J. Ryan of Stanford University and of the services rendered by Messrs. E. N. D'Oyly and Geo. L. Greves in verifying the various mathematical steps and numerical illustrations.

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BERKELEY, CALIFORNIA,
April 27, 1918.

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CHAPTER I

PHENOMENA IN CONTINUOUS CURRENT TRANSMISSION OVER NON-LEAKY LINES

1. The Simple Direct Current Circuit. — The simplest conception of a transmission circuit is that in which an unvarying current is transmitted over a system in which only the ohmic resistance of the conductors need be considered. By taking into account only the resistance of the conductors, the tacit assumption is made that between any two points which constitute the ends of the transmission system under discussion (either as actual termini or as points at which discontinuities are introduced) there is no "leakage" of current from one conductor or "side" of the system to the other. The electric current is therefore the same in magnitude at both ends of such a system, while the potential difference, or voltage, between the conductors changes from point to point because of the electromotive forces consumed by the resistance and current flow.

2. Non-uniform Conductors. — The general case for the above is that in which the resistance of the conductors per unit length along the system is not constant. Physically, such a condition brings to mind a circuit in which the conductors are of varying cross-section or material.

Since the cross-section or material of the conductor is varying from point to point, the resistance of the conductor per unit length along the circuit must be defined by the equation

$$r = \frac{dR}{dl}, \tag{1}$$

in which dR is the actual resistance of the circuit included within the length dl. This expression can be made to include both wires of the system if they are parallel and therefore of equal lengths.

Let A = cross-sectional area of conductor.

ρ = specific resistance of conductor material.

Then

$$dR = dR_1 + dR_2 = \frac{\rho_1\,dl}{A_1} + \frac{\rho_2\,dl}{A_2},$$

$$r = \frac{dR}{dl} = \frac{\rho_1}{A_1} + \frac{\rho_2}{A_2}, \tag{2}$$

in which the subscripts 1 and 2 refer to the separate wires or conductors. Unless otherwise specified, r will be understood to be defined by (2), thereby including both conductors. Since r varies along the line, in order that the problem be determinate its value must be given as a function of distance, from one end of the line say, and therefore

$$r = f(l).$$

Obviously, from (2) the total line resistance of the system is

$$R_t = \int_0^{l_0} r\,dl = \int_0^{l_0} f(l)\,dl. \tag{3}$$

If E = voltage or potential difference between the conductors at any point, the gradient of potential difference at this point is defined by

$$F = \frac{dE}{dl} = \text{gradient of potential difference.} \tag{4}$$

If I_0 = current in the conductors and the distance l be counted positively from the load end towards the source of energy,

$$F = \frac{dE}{dl} = rI_0 = I_0 f(l). \tag{5}$$

The integral of (5) gives the expression for line voltage at any point,

$$E = \int I_0 f(l)\,dl = I_0 R + C.$$

R = line resistance between the point in question and the load, and therefore the integration constant C is seen to be equal to the voltage at the load, E_0.

$$E = E_0 + RI_0, \tag{6}$$

and at the supply end, where $l = l_0$,

$$E = E_0 + R_t I_0. \tag{7}$$

In short, it is seen that even though the resistance be distributed in a non-uniform manner, it is entirely unnecessary to consider anything except the total value of line resistance,

$$R_t = \int_0^{l_0} r\,dl,$$

unless the manner of distribution of the various electrical quantities throughout the line length is desired. This simplicity arises from the fact that the line current is uniform or constant in value throughout the line length, and therefore all integrals reduce to expressions involving only the total resistance. Though very simple, this matter is one of fundamental importance, for it is due to departures from this condition that the more complicated solutions become necessary.

3. Transmission Phenomena. — Utilizing the value of total line resistance R_t, the entire discussion of such a transmission line is nothing more than that of a simple electrical circuit.

E_0 = voltage at load or receiving end.
I_0 = current throughout the system.
R_t = total line resistance.

Then

$$E = E_0 + R_t I_0 = \text{generator voltage.} \tag{8}$$

$$P_0 = E_0 I_0 = \text{power at load, in watts.} \tag{9}$$

$$P = E I_0 = P_0 + R_t I_0^2 = \text{power at generator.} \tag{10}$$

$$P' = R_t I_0^2 = \text{total line loss.} \tag{11}$$

$$\text{Eff.} = \frac{P_0}{P} = \frac{E_0}{E} = \text{efficiency of transmission.} \tag{12}$$

The voltage regulation of a transmission line is defined as

$$\frac{\text{rise in receiving end voltage from load to no load}}{\text{load voltage}},$$

the generator voltage being supposed to remain constant after throwing off the load. Under load conditions, by equation (8) the receiving end voltage is

$$E_0 = E - R_t I_0,$$

and at no load, therefore, $E_0 = E$. The rise in voltage from load to no load is R_tI_0.

$$\text{Reg.} = \frac{R_tI_0}{E - R_tI_0} = \text{regulation at load } I_0. \tag{13}$$

As will be noted, equation (13) is not that of a straight line with I_0 as independent variable if the generator voltage E be supposed constant for all loads. If, however, the generator voltage be adjusted with changing load so that the receiver voltage E_0 remains constant, then the regulation will obviously be given by

$$\text{Reg.} = \frac{R_tI_0}{E_0}, \text{ for constant } E_0.$$

With a constant generator voltage E, it is interesting to notice the change in the power at the receiver, P_0, with a change in current I_0.

$$P_0 = E_0I_0 = (E - R_tI_0)\, I_0 = EI_0 - R_tI_0^2. \tag{14}$$

From this equation it is seen that P_0 is a quadratic function of I_0, and therefore for any particular value of P_0 there are two possible values of current I_0. Also, for certain values of P_0 the expressions for I_0 become complex imaginaries, indicating physically that the value of P_0 for which the solution was made was larger than the line in question is able to transmit at the assumed value of generator voltage. This is, of course, for positive values of P_0.

The maximum amount of power which may be transmitted over the line of resistance R_t, at a generator voltage E, is to be obtained from (14). By differentiation

$$\frac{dP_0}{dI_0} = E - 2\,R_tI_0 = 0, \text{ for maximum } P_0, \tag{15}$$

thus

$$I_0 = \frac{E}{2\,R_t}, \text{ for maximum } P_0, \tag{16}$$

and the maximum power becomes, by substituting (16) in (14),

$$(P_0)_{\text{max.}} = \frac{E^2}{4\,R_t}. \tag{16a}$$

For the current value determined by (16) it is obvious that the load resistance is equal to the line resistance, $R_0 = R_t$, which is the

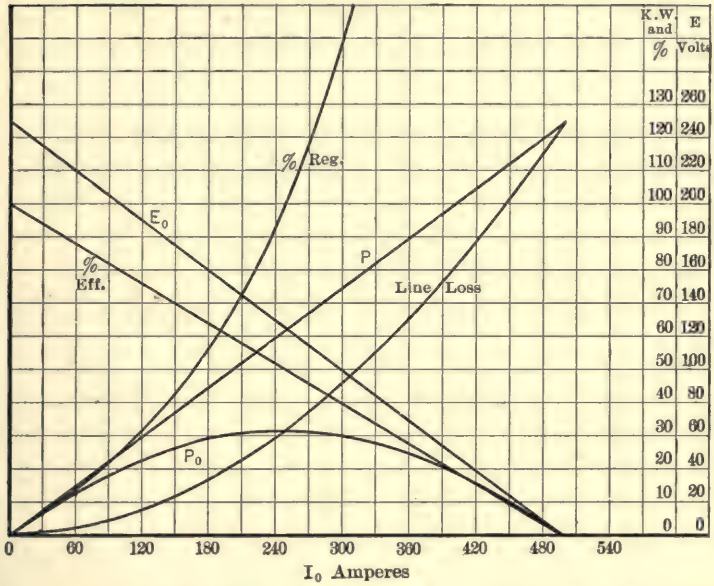

FIG. 1. — Direct Current Transmission Line Characteristics. Line Resistance, 0.50 ohm. Generator Voltage Constant, 250 Volts.

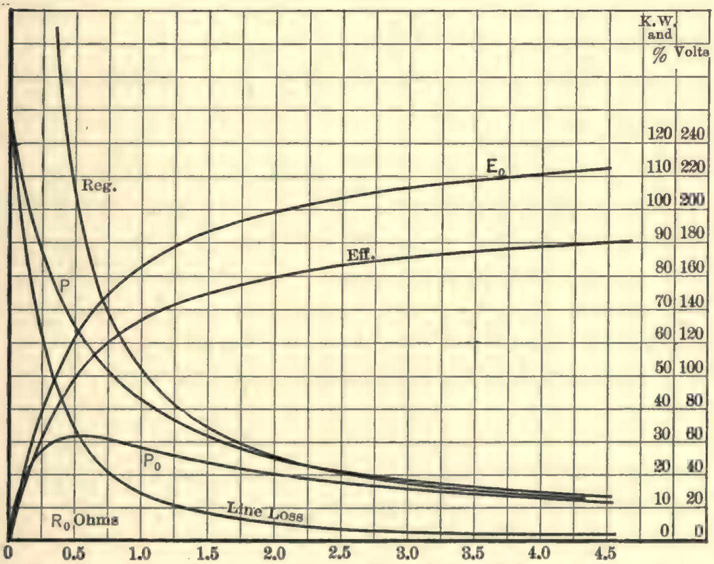

FIG. 2. — Direct Current Transmission Line Characteristics. Line Resistance, 0.50 ohm. Generator Voltage Constant, 250 Volts.

familiar relation between internal or line resistance, and load resistance, for a maximum power at the receiver.

Although in the preceding all quantities have been expressed in terms of the load current and load voltage, it may be convenient to introduce as independent variable the load resistance R_0.

Thus, substituting $E_0 = R_0 I_0$ in equation (12) for line efficiency, and simplifying the expression,

$$\text{Eff.} = \frac{R_0}{R_0 + R_t}, \tag{17}$$

which is an expression independent of either the voltage or current.

Similarly, in equation (13) for line regulation,

$$\text{Reg.} = \frac{R_t}{R_0}. \tag{18}$$

For power, since

$$I_0 = \frac{E}{R_0 + R_t} \quad \text{and} \quad E_0 = R_0 I_0,$$

$$P = EI_0 = \frac{E^2}{R_0 + R_t} = \text{power at generator.} \tag{19}$$

$$P_0 = R_0 I_0^2 = \frac{R_0}{(R_0 + R_t)^2} E^2 = \text{power at load.} \tag{20}$$

For the case of constant generator voltage, Fig. 1 shows a number of curves plotted to I_0 as independent variable, and Fig. 2 shows the curves for the same system plotted with the load resistance R_0 as independent variable.

The above discussion covers the more important phenomena in a simple direct-current transmission where line resistance only need be considered. The equations are intended to cover only the case of a line supplied with power at one end and loaded at the other. Obviously, in the practical distribution of power the simple conditions mentioned are not often realized. Such problems as the design of a railway feeder to supply power to an interurban railway, for instance, involve much more than a consideration of the simple matters mentioned above. There will be no attempt made here to take up such matters as distribution problems, particularly as there are already a number of publications dealing exclusively with the practice and economics of power distribution.

CHAPTER II

DIRECT-CURRENT LINE WITH LEAKAGE

4. General. — Under this head will be discussed the case of a line whose conductors are of uniform resistance and between which there exists a uniformly distributed conductance, or means of leakage. A portion of the following has already been presented in numerous works, but it is included here for the sake of completeness.

The following discussion is based entirely upon the use of hyperbolic functions, the natural means of expression for the phenomena occurring in systems of the character named. In fact, throughout the entire discussion of transmission systems, the use of hyperbolic functions leads to an elegance of treatment which cannot be attained by any other means.

5. Fundamental Differential Equation and Solution. — Let

r = resistance of both conductors per unit length of line, ohms.
g = conductance from wire to wire per unit length of line, mhos.
l = distance counted positively towards energy supply.
E = line voltage at any point at distance, l.
I = line current at any point at distance, l.

The resistance of an element of line of length dl is $r\,dl$, and the conductance between wires is $g\,dl$.

Therefore, taking differentials

$$\left.\begin{aligned} dI &= Eg\,dl, & dE &= Ir\,dl, \\ \frac{dI}{dl} &= Eg, & \frac{dE}{dl} &= Ir. \end{aligned}\right\} \qquad (1)$$

Differentiating again with respect to l,

$$\left.\begin{aligned} \frac{d^2I}{dl^2} &= g\,\frac{dE}{dl}, \\ \frac{d^2E}{dl^2} &= r\,\frac{dI}{dl}. \end{aligned}\right\} \qquad (2)$$

Substituting the values of the first derivatives from (1) in (2),

$$\left.\begin{aligned} \frac{d^2I}{dl^2} - rgI &= 0, \\ \frac{d^2E}{dl^2} - rgE &= 0. \end{aligned}\right\} \qquad (3)$$

The two equations in (3) are identical in form, and are of the general type of linear, second order differential equation

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0,$$

of which the general integral is

$$y = C_1 \epsilon^{m_1 x} + C_2 \epsilon^{m_2 x},$$

in which m_1 and m_2 are the two roots of the quadratic

$$a_0 m^2 + a_1 m + a_2 = 0,$$

and C_1 and C_2 are integration constants.

In (3), the particular case under discussion,

$$a_0 = 1, \quad a_1 = 0, \quad \text{and} \quad a_2 = -rg,$$

which gives

$$\left.\begin{aligned} m^2 - rg &= 0, \\ m_1 &= +\sqrt{rg}, \\ m_2 &= -\sqrt{rg}. \end{aligned}\right\} \qquad (4)$$

The solutions of (3) are therefore

$$\left.\begin{aligned} E &= A_1 \epsilon^{\sqrt{rg}\,l} + A_2 \epsilon^{-\sqrt{rg}\,l}, \\ I &= B_1 \epsilon^{\sqrt{rg}\,l} + B_2 \epsilon^{-\sqrt{rg}\,l}, \end{aligned}\right\} \qquad (5)$$

where A_1, A_2, B_1, and B_2 are integration constants to be determined by initial conditions.

6. Determination of Integration Constants. — As initial conditions, let the current and voltage at the load, or receiving end, be designated by I_0 and E_0, respectively. At this end, by the arbitrary choice of positive direction along the line $l = 0$ and the exponential terms in (5), reduce to unity.

To evaluate the integration constants, differentiate the first of

(5) and substitute the result in the second equation of (1) for the point $l = 0$. Thus,

$$\left.\frac{dE}{dl}\right]_{l=0} = \sqrt{rg}A_1 - \sqrt{rg}A_2 = rI_0. \tag{6}$$

Also, by (5),

$$\left.E\right]_{l=0} = A_1 + A_2 = E_0.$$

The solution of the two simultaneous equations (6) yields

$$\left.\begin{aligned} A_1 &= \frac{1}{2}\left(E_0 + I_0\sqrt{\frac{r}{g}}\right), \\ A_2 &= \frac{1}{2}\left(E_0 - I_0\sqrt{\frac{r}{g}}\right). \end{aligned}\right\} \tag{7}$$

By symmetry, or by proceeding in an exactly similar manner,

$$\left.\begin{aligned} B_1 &= \frac{1}{2}\left(I_0 + E_0\sqrt{\frac{g}{r}}\right), \\ B_2 &= \frac{1}{2}\left(I_0 - E_0\sqrt{\frac{g}{r}}\right), \end{aligned}\right\} \tag{8}$$

and therefore, in the general integral (5),

$$\left.\begin{aligned} E &= \frac{1}{2}\left[\left(E_0 + I_0\sqrt{\frac{r}{g}}\right)\epsilon^{\sqrt{rg}\,l} + \left(E_0 - I_0\sqrt{\frac{r}{g}}\right)\epsilon^{-\sqrt{rg}\,l}\right], \\ I &= \frac{1}{2}\left[\left(I_0 + E_0\sqrt{\frac{g}{r}}\right)\epsilon^{\sqrt{rg}\,l} + \left(I_0 - E_0\sqrt{\frac{g}{r}}\right)\epsilon^{-\sqrt{rg}\,l}\right]. \end{aligned}\right\} \tag{9}$$

Equation (9) is complete, in that it expresses the voltage and current at any point explicitly in terms of the line constants and the voltage and current at the load end. In general, it will be found that it is most convenient to so utilize the load quantities as independent variables, though for some purposes other procedures may be more desirable. This matter will be discussed later, in connection with the performance of lines carrying alternating currents.

7. Solutions in Terms of Hyperbolic Functions. — Although complete, equation (9) is not in very convenient form for computation. By combining terms, and letting

$$v = \sqrt{rg}, \tag{10}$$

a better form results.

$$\left.\begin{aligned} E &= \frac{1}{2}(\epsilon^{vl} + \epsilon^{-vl})\,E_0 + \sqrt{\frac{r}{g}}\,\frac{1}{2}(\epsilon^{vl} - \epsilon^{-vl})\,I_0, \\ I &= \frac{1}{2}(\epsilon^{vl} + \epsilon^{-vl})\,I_0 + \sqrt{\frac{g}{r}}\,\frac{1}{2}(\epsilon^{vl} - \epsilon^{-vl})\,E_0. \end{aligned}\right\} \tag{11}$$

The parentheses are immediately seen to constitute the two hyperbolic functions, cosh vl and sinh vl, which will be utilized therefor. The equations then become

$$\left.\begin{aligned} E &= E_0 \cosh vl + I_0\sqrt{\frac{r}{g}}\sinh vl, \\ I &= I_0 \cosh vl + E_0\sqrt{\frac{g}{r}}\sinh vl. \end{aligned}\right\} \tag{12}$$

$v = \sqrt{rg}$ may be considered as one of the fundamental line constants, as well as the quantity $\sqrt{\frac{r}{g}}$ or its reciprocal $\sqrt{\frac{g}{r}}$. In fact, these two quantities completely specify the characteristics of a direct-current transmission circuit when operating under steady conditions, and as far as computation purposes are concerned they constitute a more convenient means of specification than the simple quantities r and g, though either set is easily obtained from the other.

For future convenience in writing, the line constants will be expressed in either one of the alternative forms, as follows:

$$\left.\begin{aligned} &r = \text{line resistance per unit length}, \\ &g = \text{leakage conductance per unit length}, \\ \text{or}\qquad &v = \sqrt{rg}, \\ \text{and}\qquad &z = \sqrt{\frac{r}{g}}, \qquad y = \frac{1}{z} = \sqrt{\frac{g}{r}}. \end{aligned}\right\} \tag{13}$$

The fundamental line equations, on which subsequent developments are based, are then

$$\left.\begin{aligned} E &= E_0 \cosh vl + I_0 z \sinh vl, \\ I &= I_0 \cosh vl + E_0 y \sinh vl. \end{aligned}\right\} \tag{14}$$

8. Particular Forms of Solutions for $g = 0$ and for $r = 0$. If either r or g becomes zero, then equations (12) or (14) assume an indeterminate form, which may be evaluated either by the standard

methods of calculus for the evaluation of indeterminate forms, or by a consideration of the actual physical phenomena occasioned by one of these constants becoming equal to zero.

$$\left.\begin{aligned} &\text{When } g = 0, \\ &\qquad E = E_0 + I_0 rl, \\ &\qquad I = I_0 \quad \text{(discussed in Chapter I).} \\ &\text{When } r = 0, \\ &\qquad E = E_0 \\ &\qquad I = I_0 + E_0 gl. \end{aligned}\right\} \tag{15}$$

9. Load-end Quantities in Terms of Generator-end Quantities. — It is often desirable to express the quantities E_0 and I_0 in terms of E and I. It is apparent that such a solution might be obtained from (14) by substituting $-l$ for l, which would be equivalent to counting distance as positive in the direction of energy flow. To leave no doubt as to the validity of such a procedure, solve the two equations (14) simultaneously for E_0 and I_0. The result is

$$E_0 = \frac{E \cosh vl - Iz \sinh vl}{\cosh^2 vl - \sinh^2 vl},$$

$$I_0 = \frac{I \cosh vl - Ey \sinh vl}{\cosh^2 vl - \sinh^2 vl}.$$

The denominator is seen to be equal to unity, and therefore

$$\left.\begin{aligned} E_0 &= E \cosh vl - Iz \sinh vl, \\ I_0 &= I \cosh vl - Ey \sinh vl. \end{aligned}\right\} \tag{16}$$

Other combinations derived from the original equation (14) are of interest and importance. The load at the receiving end may be represented by an equivalent resistance, or conductance,

$$\left.\begin{aligned} &R_0 = \frac{E_0}{I_0}, \quad G_0 = \frac{1}{R_0} = \frac{I_0}{E_0}, \\ &E_0 = R_0 I_0, \quad I_0 = G_0 E_0, \end{aligned}\right\} \tag{17}$$

which values when substituted in (14) give

$$\left.\begin{aligned} &\left.\begin{aligned} E &= R_0 I_0 \cosh vl + I_0 z \sinh vl, \\ I &= I_0 \cosh vl + I_0 R_0 y \sinh vl, \end{aligned}\right\} a \\ &\left.\begin{aligned} E &= E_0 \cosh vl + E_0 G_0 z \sinh vl, \\ I &= G_0 E_0 \cosh vl + E_0 y \sinh vl, \end{aligned}\right\} b \end{aligned}\right\} \tag{18}$$

from which

$$\left.\begin{aligned} E_0 &= \frac{I}{G_0 \cosh vl + y \sinh vl}, \\ I_0 &= \frac{E}{R_0 \cosh vl + z \sinh vl}, \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} E_0 &= \frac{E}{\cosh vl + G_0 z \sinh vl}, \\ I_0 &= \frac{I}{\cosh vl + R_0 y \sinh vl}. \end{aligned}\right\} \tag{20}$$

10. Voltage Regulation. — At this point the equations for voltage regulation may be developed. The regulation is defined in the same way as was done in section 3, Chapter I.

By equation (14), the generator voltage, which remains constant on the removal of the load I_0, is

$$E = E_0 \cosh vl + I_0 z \sinh vl.$$

At no load, the conductance $G_0 = 0$, and equation (20) serves to determine the value of receiver voltage at no load with the voltage E at the generator. Let

$$E' = \text{no load receiver voltage;}$$

then by (20) for $G_0 = 0$

$$\begin{aligned} E' &= \frac{E}{\cosh vl} = \frac{E_0 \cosh vl + I_0 z \sinh vl}{\cosh vl} \\ &= E_0 + I_0 z \tanh vl. \end{aligned} \tag{21}$$

$$\text{Reg.} = \frac{E' - E_0}{E_0} = \frac{I_0}{E_0} z \tanh vl. \tag{22}$$

For any given line, $z \tanh vl$ is a constant, and therefore for such a condition of operation that the load voltage E_0 is maintained constant by adjustment of the generator voltage, the curve of voltage regulation when plotted to load current as abscissæ will be a straight line, the same as for the simple case in which line leakage was not considered. The magnitude of the regulation will differ.

The case parallel to that illustrated by equation (13), Chapter I,

where the generator voltage is supposed to remain constant at all loads will now be considered.

E = constant generator voltage,
I_0 = load current.

Solving (14) for the value of E_0 in terms of E and I_0,

$$E_0 = \frac{E - I_0 z \sinh vl}{\cosh vl} = E \operatorname{sech} vl - I_0 z \tanh vl. \qquad (23)$$

By (20) the no-load receiver voltage is $E' = E \operatorname{sech} vl$.

$$\text{Reg.} = \frac{E' - E_0}{E_0} = \frac{I_0 z \tanh vl}{E \operatorname{sech} vl - I_0 z \tanh vl}$$
$$= \frac{I_0}{Ey \operatorname{cosech} vl - I_0}. \qquad (24)$$

Equation (24), as in the case of equation (13), Chapter I, for the line without leakage, is not that of a straight line but of a curve passing through the origin and reaching a value of infinity at a value of $I_0 = Ey \operatorname{cosech} vl$, which is the value of I_0 occurring when the load end of the line is short-circuited. This may be verified from equation (19) by placing $R_0 = 0$.

11. Power Relations in General. — At the load end of the line the power received is

$$P_0 = E_0 I_0 \text{ watts}.$$

At any other point the power is given by

$$P = EI,$$

E and I being given by (14). Multiplying the two expressions in (14),

$$P = EI = E_0 I_0 \cosh^2 vl + E_0 I_0 \sinh^2 vl$$
$$+ E_0^2 y \sinh vl \cosh vl + I_0^2 z \sinh vl \cosh vl. \qquad (25)$$

By the hyperbolic reduction formulæ in the Appendix, this reduces to

$$P = P_0 \cosh 2\,vl + \tfrac{1}{2}(I_0^2 z + E_0^2 y) \sinh 2\,vl. \qquad (26)$$

The total line loss is $P' = P - P_0$. Utilizing this relation in (25),

$$P' = P_0 (\cosh^2 vl + \sinh^2 vl - 1)$$
$$+ E_0^2 y \sinh vl \cosh vl + I_0^2 z \sinh vl \cosh vl;$$
$$P' = P_0^2 \sinh^2 vl + (E_0^2 y + I_0^2 z) \sinh vl \cosh vl. \qquad (27)$$

For lines with very small losses it is desirable to calculate the power loss directly by (27) since a much greater accuracy may be secured in this way than by calculating P and then numerically subtracting P_0, for in this latter case the difference between two large quantities of nearly the same value must be taken.

In presenting the formulæ as developed, it is not expected that they are to be applied indiscriminately to any and all cases, for though they are rigid, and will therefore yield accurate results wherever they are applied, it may not be at all necessary to introduce the degree of refinement that their use yields. Individual cases must be treated individually, if an efficient use is to be made of the material at hand. Obviously, there is little sense in calculating line phenomena by rigid formulæ to a degree of accuracy yielded by six-place logarithms, say, when the line constants on which the computations are based are perhaps in error by as much as two or three per cent. Such matters must be left to the judgment of the individual, but they cannot detract from the desirability and need of preparing and presenting rigid expressions.

12. Maximum Power. — Following the scheme adopted in section 3, Chapter I, the maximum value of power which can be transmitted over a leaky line by a generator of constant voltage, E, will now be investigated. To do this, E_0 and P_0 will be expressed in terms of E and I_0, and since E is constant the value of I_0 which will give a maximum P_0 can be found.

By (23) $$E_0 = E \operatorname{sech} vl - I_0 z \tanh vl,$$

$$P_0 = E_0 I_0 = I_0 E \operatorname{sech} vl - I_0^2 z \tanh vl. \tag{28}$$

For a maximum, the first derivative of P_0 with respect to I_0 must be equal to zero, and the second derivative must be negative.

$$\frac{dP_0}{dI_0} = E \operatorname{sech} vl - 2\, I_0 z \tanh vl = 0 \text{ for max.} \tag{29}$$

$$\frac{d^2P_0}{dI_0^2} = -2\, z \tanh vl, \text{ which is negative.}$$

Thus, by (29), the load current for maximum power is

$$I_0 \Big]_{P_0 = \max.} = \frac{E \operatorname{sech} vl}{2\, z \tanh vl} = \frac{E}{2\, z \sinh vl}. \tag{30}$$

Substituting (30) in (28) and collecting terms,

$$P_0 \text{ max.} = \frac{E^2}{4\,z \sinh vl \cosh vl} = \frac{E^2}{2\,z \sinh 2\,vl}. \tag{31}$$

It is interesting to compare the expression (31) for the maximum power which can be transmitted over a leaky line with the expression for the maximum power which can be transmitted over a non-leaky line, which is, by (16a), Chapter I,

$$P_0 \text{ max.}\Big]_{g=0} = \frac{E^2}{4\,R_t}. \tag{32}$$

It is evident that $4\,z \sinh vl \cosh vl = 2\,z \sinh 2\,vl$ replaces the quantity $4\,R_t$ when there is leakage present. By introducing into (31) the condition that $g = 0$, the expression should reduce to (32). This will be shown to be true.

Expanding $2\,z \sinh 2\,vl$ into a series, using the values of z and v by (15),

$$\begin{aligned} 2\,z \sinh 2\,vl &= 2\sqrt{\frac{r}{g}}\left[2\sqrt{rgl} + \frac{1}{\lfloor 3}(2\sqrt{rgl})^3 + \cdots\right] \\ &= 4\,rl + \tfrac{8}{3}\,r^2gl^3 + \cdots. \end{aligned}$$

Now, all terms except the first contain g, and therefore for $g = 0$, $2\,z \sinh 2\,vl$ reduces to $4\,rl$, which is identical with the denominator of (32), since

$R_t = rl =$ total line resistance if uniformly distributed.

13. Efficiency. — In the non-leaky line, by equation (17), Chapter I, it is seen that for loads approaching zero (R_0 approaching infinity) the efficiency approaches unity in value. Such is not the case for the leaky line, for it will be shown here that there is a definite limit to the possible value of line efficiency, no matter what the condition of loading may be.

Using equation (26),

$$\begin{aligned} \text{Eff.} = \frac{P_0}{P} &= \frac{E_0I_0}{E_0I_0 \cosh 2\,vl + \frac{1}{2}(zI_0^2 + yE_0^2)\sinh 2\,vl} \\ &= \frac{1}{\cosh 2\,vl + \dfrac{1}{2}\left(z\dfrac{I_0}{E_0} + y\dfrac{E_0}{I_0}\right)\sinh 2\,vl}. \end{aligned} \tag{33}$$

From (33) it is apparent that the efficiency is in no way dependent upon the absolute magnitude of either E_0 or I_0 but rather

upon the ratio of the two, that is, upon R_0 or G_0. Obviously, also, the maximum efficiency will occur for such a value of this ratio as will make the term

$$u = z\frac{I_0}{E_0} + y\frac{E_0}{I_0} = \text{a minimum.}$$

Letting $G_0 = \frac{I_0}{E_0}$, and remembering that $y = \frac{1}{z}$, we have

$$u = zG_0 + \frac{1}{zG_0}.$$

For a minimum,

$$\frac{du}{dG_0} = z - \frac{1}{zG_0^2} = 0,$$

from which

$$z^2 = \frac{1}{G_0^2} = R_0^2, \quad z = \pm R_0.$$

In the double sign, only the positive can have a physical significance, and by investigating the value of the second derivative it can be shown that this sign gives a minimum to the value of u.

Thus, for a maximum efficiency,

$$\left.\begin{aligned} E_0 &= I_0 z, \\ I_0 &= E_0 y. \end{aligned}\right\} \tag{34}$$

Substituting the ratio of E_0 to I_0 as given by (34) in equation (33), the maximum efficiency which may be attained is

$$\text{Max. Eff.} = \frac{1}{\cosh 2\,vl + \sinh 2\,vl} = \epsilon^{-2vl}. \tag{35}$$

The next and final step in this development will be to express the load current for which maximum efficiency occurs in terms of the generator voltage E, which is supposed to be constant.

Introducing (23) into (34), for maximum efficiency,

$$E \operatorname{sech} vl - I_0 z \tanh vl = zI_0,$$

from which

$$I_0\Big]_{\text{Eff.} = \text{max.}} = \frac{E \operatorname{sech} vl}{z\,(1 + \tanh vl)} = \frac{E}{z}\epsilon^{-vl}. \tag{36}$$

Comparing (30) and (36) it is seen that the maximum efficiency does not occur coincidently with the maximum power at the load when the generator voltage is constant.

All of the above formulæ involving hyperbolic functions could be expanded into series, thereby furnishing approximate solutions, but it is not considered at all necessary to include such developments here. The above formulæ, involving as they do only hyperbolic functions of real or non-imaginary variables, are so conveniently used numerically that to introduce series expressions leads to needless complexity. Neither does it appear necessary to introduce various approximate methods of solution, such as assuming that the entire leakage may be concentrated at the middle of the line, or the still closer approximation of placing one-sixth of the leakage at each end and two-thirds in the middle of the line. The solutions based on such assumptions are approximations, which in reality amount to using only a limited number of terms in the series expansions of the hyperbolic functions. If at any time tables of such functions are not available, it is more convenient to introduce the series for the evaluation of the hyperbolic functions than to resort to an approximate solution based on the above mentioned assumptions.

14. Numerical Illustration. — Before proceeding further with theoretical developments, a numerical illustration of the foregoing will be given.

Consider a line with the following constants per mile of length.

$$r = 50.0 \text{ ohms.} \qquad g = 2 \times 10^{-4} \text{ mhos.}$$

$$v = \sqrt{rg} = 0.1000. \quad z = \sqrt{\frac{r}{g}} = 500.0. \quad y = \sqrt{\frac{g}{r}} = 0.00200.$$

Let the line length be $l = 15.0$ miles.

Then $vl = 1.500$.

(a) What voltage would be required at the sending end to give a current of 0.050 amperes through a resistance of 200.0 ohms at the receiving end?

$$I_0 = 0.050, \qquad R_0 = 200.0, \qquad E_0 = R_0 I_0 = 10.00.$$

By (14), $E = 10.0 \cosh 1.50 + 0.050 \times 500 \sinh 1.50$
$= 10.0 \times 2.35241 + 25 \times 2.12928 = 76.7561$ volts.

(b) What is the generator current?

By (14), $I = 0.050 \cosh 1.50 + 10.0 \times 0.00200 \sinh 1.50$
$= 0.160206$ amperes.

(c) What is the voltage regulation under these conditions?

By (22), $$\text{Reg.} = \frac{0.050}{10.00}\, 500 \tanh 1.50 = 2.50 \times 0.90515$$
$$= 2.263 = 226.3 \text{ per cent.}$$

(d) For a constant generator voltage of 100.0 volts, calculations of various quantities for different values of load current are given in the following table. The equations used are numbered to correspond with the text, and are given in the final numerical form for the particular case under discussion.

(23) $$E_0 = 100 \operatorname{sech} 1.5 - 500\, I_0 \tanh 1.5$$
$$= 42.510 - 452.58\, I_0.$$

(28) $$P_0 = E_0 I_0.$$

(16) $$I = I_0 \operatorname{sech} vl + Ey \tanh vl,$$
$$= 0.42510\, I_0 + 0.181030.$$

(21) $$E' = 100 \operatorname{sech} 1.50 = 42.510.$$

(24) $$\text{Reg.} = \frac{E' - E_0}{E_0} = \frac{I_0}{0.200 \operatorname{cosech} 1.5 - I_0}$$
$$= \frac{I_0}{0.0939284 - I_0}.$$
$$P = 100\, I.$$
$$\text{Efficiency} = \frac{P_0}{P}.$$
$$P' = \text{line loss} = P - P_0.$$

TABLE I

CALCULATION OF PERFORMANCE OF LEAKY LINE

I_0	0.0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
$452.58\, I_0$	0.0	4.526	9.052	13.577	18.103	22.629	27.155	31.681	36.206	40.732
E_0	42.510	37.984	33.458	28.933	24.407	19.881	15.355	10.829	6.304	1.778
P_0	0.0	0.37984	0.66916	0.86799	0.97628	0.99405	0.92130	0.75803	0.50432	0.16002
$0.4251\, I_0$	0.0	0.00425	0.00850	0.01275	0.01700	0.02126	0.02551	0.02976	0.03401	0.03826
I	0.18103	0.18528	0.18953	0.19378	0.19803	0.20228	0.20654	0.21079	0.21504	0.21929
Reg.	0.0	0.11915	0.27053	0.46928	0.74172	1.13822	1.76843	2.92539	5.74366	22.9101
P	18.1030	18.5283	18.9532	19.3783	19.8034	20.2285	20.6536	21.0787	21.5038	21.9289
Eff.	0.0	0.02050	0.03531	0.04479	0.04930	0.04914	0.04461	0.03596	0.02345	0.00730
P'	18.1030	18.1485	18.2840	18.5103	18.8271	19.2345	19.7323	20.3207	20.9995	21.7689

The data obtained in the above table of calculations are shown plotted in Figs. 3 and 4.

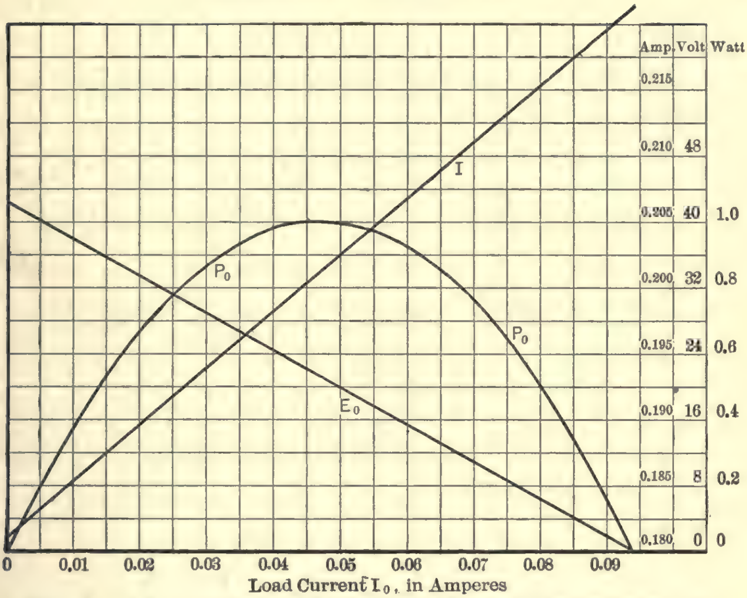

FIG. 3. — Characteristics of Leaky Direct Current Transmission Line. Generator Voltage Constant, 100 Volts.

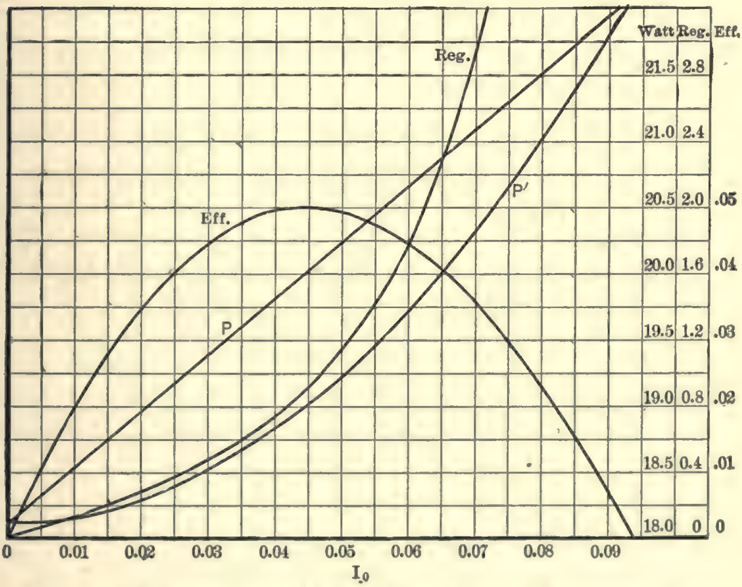

FIG. 4. — Characteristics of Leaky Direct Current Transmission Line. Generator Voltage Constant, 100 Volts.

By equation (30) the maximum P_0 occurs for $I_0 = 0.0469642$ amperes, which may be noted on the plotted curve and is, by (31), 0.99821 watts.

One more illustrative example will be considered.

Take a line with constants, $r = 5.0$ ohms, and $g = 2.0 \times 10^{-5}$ mhos, per mile. Then

$$v = 0.0100, \quad z = 500.0, \quad \text{and} \quad y = 0.00200.$$

As load values, let $E_0 = 10{,}000$ volts and $I_0 = 10.0$ amperes. For this load, the following table gives values of generator voltage, current, power, line loss, and efficiency of transmission for various line lengths.

TABLE II

l miles	E volts	I amperes	P K.W.	P' K.W.	Eff.
0	10,000.0	10.0000	100.00	0	1.0000
20	11,207.4	14.2275	159.45	59.45	0.6272
40	12,864.4	19.0257	244.75	144.75	0.4086
60	15,038.0	24.5877	369.75	269.75	0.2705
80	17,814.9	31.1365	554.69	454.69	0.1803
100	21,306.8	38.9348	829.58	729.58	0.1205

For line lengths greater than 40 or 50 miles, the generator voltage and current, and consequently power also, are seen to increase very rapidly. The example considered is, however, an exaggerated case inasmuch as the leakage coefficient, g, is several times larger than would be encountered in an actual system which was designed to transmit power at the voltage considered. The line constants were arbitrarily selected so as to furnish results illustrating the properties of the solutions.

In both this and the former illustrative example, the line losses were of sufficient size in comparison with the total power to be obtained readily by subtracting the power at the load from that at the generator, thus furnishing no necessity for calculating the loss directly by equation (27). The above results are shown in Fig. 5.

15. Effective Resistance of Line and Determination of Constants by Measurement. — For a direct current transmission system with no leakage the effective resistance of the line and

load at the generator end, as determined by the ratio E/I, is exactly equal to the load resistance plus the line resistance. For the leaky line such is not the case, the effective resistance depend-

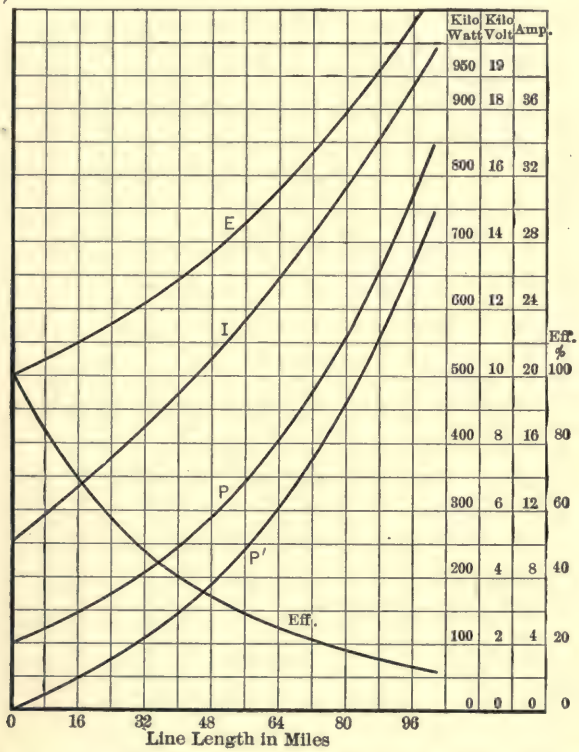

FIG. 5. — Characteristics of Leaky Direct Current Transmission Line. Load at Receiver Constant; $E_0 = 10{,}000$ volts and $I_0 = 10.0$ amperes.

ing upon the load resistance and the two line constants r and g, as well as upon the line length.

The ratio of generator voltage to generator current defines the effective resistance. Utilizing equations (18a), after canceling the common factor I_0 in numerator and denominator, one has

$$R = \frac{E}{I} = \frac{R_0 \cosh vl + z \sinh vl}{\cosh vl + R_0 y \sinh vl} \tag{37}$$

= effective resistance from generator end as would be indicated by a Wheatstone bridge.

The above equation, though complete, is scarcely as important as two special cases derivable from it. They are for the effective resistance when the load end of the line is short-circuited, and when the load end is open-circuited. The first may be derived directly from (37) by placing R_0 equal to zero, but the placing of R_0 equal to infinity (rather, allowing R_0 to increase indefinitely) for the second case would lead to an indeterminate form. Dividing both numerator and denominator of (37) by R_0, or utilizing equations (18b) gives

$$R = \frac{E}{I} = \frac{\cosh vl + G_0 z \sinh vl}{G_0 \cosh vl + y \sinh vl}, \tag{38}$$

from which, by placing $G_0 = 0$, the desired result is obtained. Thus, from (37) for $R_0 = 0$,

$$R' = z \tanh vl = \text{short-circuit resistance}; \tag{39}$$

and from (38) for $G_0 = 0$,

$$R'' = z \coth vl = \text{open-circuit resistance}. \tag{40}$$

For infinite values of line length, the two expressions (39) and (40) as well as (38) approach the value of z, since the hyperbolic tangent and cotangent both approach unity for continuously increasing values of argument. Thus, for infinite line length, the terminal or load resistance has no influence upon the effective generator-end resistance of the system, which is then equal to

$$z = \sqrt{\frac{r}{g}} = \text{infinite line resistance}. \tag{41}$$

Equations (39) and (40) are of importance in determining the values of r and g from actual measurement of line resistance. There are two quantities to be determined, and therefore two measurements taken under different conditions will be required. The two simplest conditions to utilize are obviously those mentioned above, viz., short-circuited and open-circuited, and from these two resistance measurements, taken either by means of a bridge or by the voltmeter-ammeter method, numerical values of R' and R'' will be furnished.

From these values of R' and R'' the values of r and g are to be determined.

Multiplying (39) and (40),

$$R'R'' = z^2 = \frac{r}{g},$$

and by division of these same equations,

$$\frac{R'}{R''} = \tanh^2 vl; \quad \sqrt{\frac{R'}{R''}} = \tanh l\sqrt{rg}.$$

Thus, knowing the line length, l,

$$\left.\begin{aligned} z &= \sqrt{\frac{r}{g}} = \sqrt{R'R''}, \\ v &= \sqrt{rg} = \frac{1}{l}\tanh^{-1}\sqrt{\frac{R'}{R''}}. \end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned} r &= vz = \frac{1}{l}\sqrt{R'R''}\tanh^{-1}\sqrt{\frac{R'}{R''}}, \\ g &= \frac{v}{z} = \frac{1}{l\sqrt{R'R''}}\tanh^{-1}\sqrt{\frac{R'}{R''}}. \end{aligned}\right\} \tag{43}$$

Equations (42) and (43) suffice to accurately and completely determine the fundamental line constants from the two measurements, R' and R'', made under the specified conditions. The above method, or its equivalent, must be used in order to determine the two fundamental line constants accurately when the total line resistance and leakage are relatively large, or to be more exact, when the product of these totals exceeds a certain definite amount depending upon the accuracy desired.

The ordinary approximation methods for determining the line constants consist in taking

$$\left.\begin{aligned} r &= \frac{1}{l}R', \qquad & R_t &= rl = R', \\ g &= \frac{1}{lR''}, \qquad & G_t &= gl = \frac{1}{R''}, \end{aligned}\right\} \tag{44}$$

which are based on the assumptions that when the line is short-circuited for measuring R' all the current traverses the entire line — that is, is of uniform value along the line length; and that when the line is open-circuited for measuring R'', the potential difference between conductors is the same throughout the length.

The approximation (44) is equivalent to assuming that in (43)

$$\tanh^{-1}\sqrt{\frac{R'}{R''}} \quad \text{may be replaced by} \quad \sqrt{\frac{R'}{R''}},$$

which will obviously introduce considerable error for values of the radical greater than two-tenths. It is also apparent, by (42), that the value of the ratio

$$\sqrt{\frac{R'}{R''}}$$

is dependent only upon the product $vl = \sqrt{R_t G_t}$.

The following table will show the magnitude of the error which the approximation method would introduce under various conditions:

TABLE III

THE INVERSE HYPERBOLIC TANGENT

$\sqrt{\frac{R'}{R''}}$	$\tanh^{-1}\sqrt{\frac{R'}{R''}} = \sqrt{R_t G_t} = vl$	$\sqrt{\frac{R'}{R''}}$	$\tanh^{-1}\sqrt{\frac{R'}{R''}} = \sqrt{R_t G_t} = vl$
0.00	0.00000	0.55	0.61839
0.05	0.05004	0.60	0.69314
0.10	0.10033	0.65	0.77530
0.15	0.15114	0.70	0.86729
0.20	0.20273	0.75	0.97295
0.25	0.25541	0.80	1.09861
0.30	0.30952	0.85	1.25615
0.35	0.36544	0.90	1.47221
0.40	0.42365	0.95	1.83180
0.45	0.48470	1.00	Infinity
0.50	0.54931	....	....

By the use of the above table the error introduced into the determination of the fundamental line constants by the use of the approximation method may be determined as soon as the measurements on which the computations are to be based are available.

16. Combinations of Leaky Lines. — In all of the preceding the discussion is applied to a line supplied with power at one end and delivering energy at the other end — a condition which permits the enumeration of practically all of the equations covering the phenomena of transmission of energy over leaky lines. The conditions named in the heading of this paragraph do not permit

of such ready and general treatment, though the general method of attack may be indicated. Assuming, of course, that each individual section of line in such a system is uniform throughout its length, the fundamental differential equations (3) and therefore the general solutions (14) still apply if the integration constants E_0 and I_0 are correctly interpreted.

17. Single Generator Supplying Two Separate Loads. — As a first special case, consider a generator of constant voltage E supplying power to two separate loads at A and B, the connecting transmission systems being leaky. Figure 6 serves as illustration. The two sections of line will be defined by the quantities,

$$\begin{array}{lllll} \text{Section 1,} & r_1,\ g_1,\ l_1, & \text{or} & z_1,\ v_1,\ l_1,\ y_1 = 1/z_1. \\ \text{Section 2,} & r_2,\ g_2,\ l_2, & \text{or} & z_2,\ v_2,\ l_2,\ y_2 = 1/z_2. \end{array}$$

The loads at A and B may, however, be specified in different ways. Suppose, for instance, that they be specified by means of equiva-

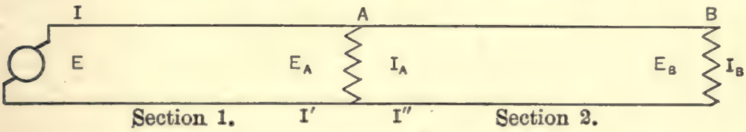

FIG. 6. — Single Generator Supplying Two Separate Loads.

lent resistances, R_a and R_b, respectively. To determine all currents and voltages in the system when the generator voltage is equal to E, proceed as follows:

The equivalent resistance R'' of the load R_b and line section 2, as measured at A, would be given by (37) as

$$R'' = \frac{E_a}{I''} = \frac{R_b \cosh v_2 l_2 + z_2 \sinh v_2 l_2}{\cosh v_2 l_2 + R_b y_2 \sinh v_2 l_2}. \tag{45}$$

The above defined equivalent resistance is in parallel with the load resistance R_a, and thus the equivalent resistance of the two combined is

$$\left.\begin{aligned} R' &= \frac{E_a}{I'} = \frac{R_a R''}{R_a + R''}, \\ G' &= 1/R'. \end{aligned}\right\} \tag{46}$$

Then immediately, by (19),

$$I' = \frac{E}{R' \cosh v_1 l_1 + z_1 \sinh v_1 l_1}. \tag{47}$$

Also,

$$\left.\begin{aligned} E_a &= R'I', \\ I_a &= \frac{E_a}{R_a}, \\ I'' &= \frac{E_a}{R''} = I' - I_a. \end{aligned}\right\} \tag{48}$$

Again, by equation (19),

$$\left.\begin{aligned} I_b &= \frac{E_a}{R_b \cosh v_2 l_2 + z_2 \sinh v_2 l_2}, \\ E_b &= R_b I_b. \end{aligned}\right\} \tag{49}$$

Finally, at the generator, by equation (12),

$$I = I' \cosh v_1 l_1 + E_a y_1 \sinh v_1 l_1. \tag{50}$$

Equations (45) to (50), which apply to the particular case under discussion, are arranged in convenient form for computation. They are not entirely independent inasmuch as the results obtained from one equation are introduced into one following.

It is of interest to note the quantities which may be specified at will in such a transmission system. Assuming the line constants as fixed and the loading points A and B established, we may specify, for example,

(1) E_b, I_b, and I_a,
(2) E_b, E_a, and I_a,
(3) R_a, R_b, and E,

but not such combinations as

(4) E_a, E_b, and I_b,
(5) E_a, I_b, and I_a,

for in the latter it is readily seen that the quantities are not independent.

It is impossible to treat here an extended number of special cases, but the example just given may serve as a help in deciding upon a method of attack.

18. Single Load Supplied from Two Power Sources. — A different form of problem is that in which a single load is supplied from two sources of power over lines possessing leakage, as shown in Fig. 7.

Again, the problem may be stated in several different ways, some of which lead to impossible solutions. For example, with

fixed generator voltage and a short-circuit at the point of loading, the current I_0 would be the sum of the two currents supplied by the two separate short-circuited lines, which, by equation (19) when $R_0 = 0$, would be

$$E_1 y_1 \operatorname{cosech} v_1 l_1 + E_2 y_2 \operatorname{cosech} v_2 l_2.$$

It would be absurd, with fixed generator voltages, to specify a load current I_0 greater than the above which could be supplied

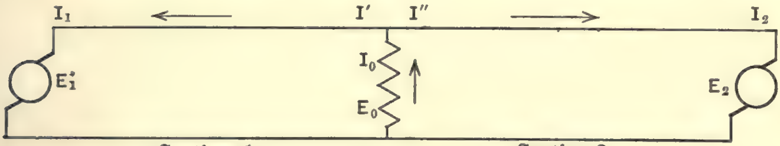

FIG. 7. — Single Load Supplied by Two Generators.

over a short-circuit. The same is true in the discussion of a similar system without leakage.

As a concrete case, let the resistance of the load, $R_0 = \frac{E_0}{I_0}$, be specified. All electrical quantities for given values of E_1 and E_2 are to be determined. Let the convention as to positive direction in each circuit be as indicated by the arrows. Then, with the notation shown in the diagram,

$$\left.\begin{aligned} I_0 &= I' + I'', \\ E_0 &= R_0 I' + R_0 I''. \end{aligned}\right\} \qquad (51)$$

The two currents I' and I'' will be taken as the two unknown quantities whose determination constitutes the main feature of the solution of this problem. By equation (14),

$$E_1 = E_0 \cosh v_1 l_1 + I' z_1 \sinh v_1 l_1,$$
$$E_2 = E_0 \cosh v_2 l_2 + I'' z_2 \sinh v_2 l_2.$$

Substituting for E_0 its value as given by (51) and collecting terms,

$$\left.\begin{aligned} (R_0 \cosh v_1 l_1 + z_1 \sinh v_1 l_1)\, I' + (R_0 \cosh v_1 l_1)\, I'' &= E_1, \\ (R_0 \cosh v_2 l_2)\, I' + (R_0 \cosh v_2 l_2 + z_2 \sinh v_2 l_2)\, I'' &= E_2. \end{aligned}\right\} \qquad (52)$$

In (52) there are two simultaneous equations in the two unknown quantities I' and I'', which suffices for their determination. The solution by determinants yields

$$\left.\begin{aligned}
I' &= \frac{(R_0 \cosh v_2l_2 + z_2 \sinh v_2l_2)\, E_1 - (R_0 \cosh v_1l_1)\, E_2}{(R_0 \cosh v_1l_1 + z_1 \sinh v_1l_1)\,(R_0 \cosh v_2l_2 + z_2 \sinh v_2l_2) - R_0^2 \cosh v_1l_1 \cosh v_2l_2},\\
I'' &= \frac{(R_0 \cosh v_1l_1 + z_1 \sinh v_1l_1)\, E_2 - (R_0 \cosh v_2l_2)\, E_1}{(R_0 \cosh v_1l_1 + z_1 \sinh v_1l_1)\,(R_0 \cosh v_2l_2 + z_2 \sinh v_2l_2) - R_0^2 \cosh v_1l_1 \cosh v_2l_2},\\
I_0 &= \frac{(z_2 \sinh v_2l_2)\, E_1 + (z_1 \sinh v_1l_1)\, E_2}{(R_0 \cosh v_1l_1 + z_1 \sinh v_1l_1)\,(R_0 \cosh v_2l_2 + z_2 \sinh v_2l_2) - R_0^2 \cosh v_1l_1 \cosh v_2l_2}.
\end{aligned}\right\} \quad (53)$$

The voltage at the load is R_0I_0, which becomes, after expanding the denominator of the last equation of (53) and multiplying by R_0,

$$E_0 = R_0I_0 = \frac{(z_2 \sinh v_2l_2)\, E_1 + (z_1 \sinh v_1l_1)\, E_2}{z_2 \cosh v_1l_1 \sinh v_2l_2 + z_1 \cosh v_2l_2 \sinh v_1l_1 + \dfrac{z_1z_2}{R_0} \sinh v_1l_1 \sinh v_2l_2}. \quad (54)$$

19. Unloaded Line with Double Power Supply. — Having determined the components of the load current by (53) and the load voltage, the determination of all other quantities is accomplished with facility by the application of the several general equations applying to simple circuits. The results in equations (53) and (54) are perfectly general, holding for all values of load resistance, and they therefore lead directly to a method of determining the line current and voltage at any point when no load is present by the simple device of letting the resistance R_0 approach infinity in value.

Expanding the denominator in (53), dividing both numerator and denominator by R_0, and discarding all terms which reduce to zero on account of the infinite value of R_0, the resulting equations for the line currents become

$$\left.\begin{aligned}
I'\Big]_{R_0=\infty} &= \frac{E_1 \cosh v_2l_2 - E_2 \cosh v_1l_1}{z_2 \cosh v_1l_1 \sinh v_2l_2 + z_1 \cosh v_2l_2 \sinh v_1l_1},\\
I''\Big]_{R_0=\infty} &= \frac{E_2 \cosh v_1l_1 - E_1 \cosh v_2l_2}{z_2 \cosh v_1l_1 \sinh v_2l_2 + z_1 \cosh v_2l_2 \sinh v_1l_1},
\end{aligned}\right\} \quad (55)$$

while from (54), the voltage at the point in question is immediately

$$E_0\Big]_{R_0=\infty} = \frac{E_1z_2 \sinh v_2l_2 + E_2z_1 \sinh v_1l_1}{z_2 \cosh v_1l_1 \sinh v_2l_2 + z_1 \cosh v_2l_2 \sinh v_1l_1}. \quad (56)$$

The results expressed by equations (53) to (56) are of interest and importance. In the case of the loaded line as covered by equations (53) and (54), the distribution of current and voltage along either portion of the transmission system is conveniently

obtained by use of equation (14), counting distance from the point of loading and using the results obtained from (53) and (54) as initial quantities. Since the solutions as given are general, either positive or negative numerical values may be assigned to the electromotive forces E_1 and E_2. The meaning of the algebraic sign in a numerical solution resulting from the use of these equations follows the convention usual in any application of Kirchhoff's Laws where a direction as indicated by an arrow is arbitrarily taken as positive.

Equations (55) and (56) applied to a uniform, unloaded line serve immediately for the determination of the distribution of voltage and current. In a non-uniform or composite line containing two sections of line with different constants, the solution must be made first for the voltage and current at the junction point, and then, utilizing equation (14) by the method mentioned in the preceding paragraph, for any desired point throughout the length of either section. Of course, in the case of the uniform, unloaded line whose constants are the same throughout the entire length between the two sources of power supply, the lengths l_1 and l_2 must be so taken that their sum is equal to the total line length.

Naturally, if the line is uniform throughout, a simplified form of expression will result from either (53) and (54) or from (55) and (56), for in this case $z_1 = z_2$, and the denominators in (55) and (56), for example, reduce to a single function. For a uniform line whose total length is $l_1 + l_2$, with constants v and z per unit length,

$$\left.\begin{aligned} I'\Big]_{R_0=\infty} &= \frac{E_1 \cosh vl_2 - E_2 \cosh vl_1}{z \sinh v\,(l_1 + l_2)}, \\ I''\Big]_{R_0=\infty} &= \frac{E_2 \cosh vl_1 - E_1 \cosh vl_2}{z \sinh v\,(l_1 + l_2)}. \end{aligned}\right\} \tag{57}$$

$$E_0\Big]_{R_0=\infty} = \frac{E_1 \sinh vl_2 + E_2 \sinh vl_1}{\sinh v\,(l_1 + l_2)}. \tag{58}$$

The denominator is constant for all points selected, since $l_1 + l_2$ is equal to the total line length, and therefore constant.

20. Numerical Illustration. — A numerical illustration will be given here of the distribution of current and potential difference along a uniform, unloaded line, utilizing equations (57) and (58).

Let the total line length be 100 miles and the line constants be

$$r = 10.0, \quad g = 10^{-5}, \quad \text{per mile.}$$

Then

$$z = z_1 = z_2 = 10^3. \quad y = y_1 = y_2 = 10^{-3}.$$
$$v = v_1 = v_2 = 0.0100.$$

The denominator of (57) is $10^3 \sinh 1.00 = 1175.20$.
The denominator of (58) is $\sinh 1.00 = 1.17520$.

Let $E_1 = 100$ volts and $E_2 = 80$ volts, both positive, and therefore acting in the direction indicated by the arrows in Fig. 7. The following table contains the calculations for determining the distribution of line voltage and current.

TABLE IV

LINE WITH DOUBLE SOURCE OF POWER SUPPLY

l_1	00	10	20	30	40	50	60	70	80	90	100
l_2	100	90	80	70	60	50	40	30	20	10	00
vl_1	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
vl_2	1.00	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10	0.00
$E_2 \sinh vl_1$	0.00	8.01	16.11	24.36	32.86	41.69	50.93	60.69	71.05	82.12	94.02
$E_1 \sinh vl_2$	117.52	102.65	88.81	75.86	63.66	52.11	41.08	30.45	20.13	10.02	00.00
Sum	117.52	110.66	104.92	100.22	96.52	93.80	92.01	91.14	91.18	92.14	94.02
E_0	100.00	94.16	89.28	85.28	82.13	79.81	78.29	77.55	77.59	78.40	80.00
$E_1 \cosh vl_2$	154.31	143.31	133.74	125.52	118.55	112.76	108.11	104.53	102.01	100.50	100.00
$E_2 \cosh vl_1$	80.00	80.40	81.61	83.63	86.49	90.21	94.84	100.41	106.99	114.65	123.45
Difference	74.31	62.91	52.13	41.89	32.06	22.55	13.27	4.12	−4.98	−14.15	−23.45
$10^3 \times I'$	63.23	53.53	44.36	35.64	27.28	19.19	11.29	3.51	−4.24	−12.04	−19.95

The distribution of current and voltage along the line as determined in the above table of computations is shown graphically in Fig. 8. Naturally, with an unloaded line supplied with energy at both ends in the manner considered in this illustration, the direction of current flow reverses at a point coinciding with the minimum of line voltage.

21. General Network of Transmission Systems with Loads at Various Points. — Of such a system as mentioned very little can be said unless a large number of special cases be taken under consideration, and as even then the field of possibilities could in no way be covered, the solution of such problems, if an exact solution be desired, must be left to the ingenuity and mathematical ability of the individual. In general, unless the problem be stated in a manner at variance with the physical possibilities (for instance by specifying a load current at some point greater than could be supplied by generators of the selected voltage over the transmission systems involved), it will always be possible to obtain a

solution. Kirchhoff's Laws will in general supply a number of current equations. If the generator electromotive forces are specified and the loads represented by their equivalent resistances, the scheme adopted in arriving at equations (51) to (54) will serve in a majority of cases. The known electromotive forces are

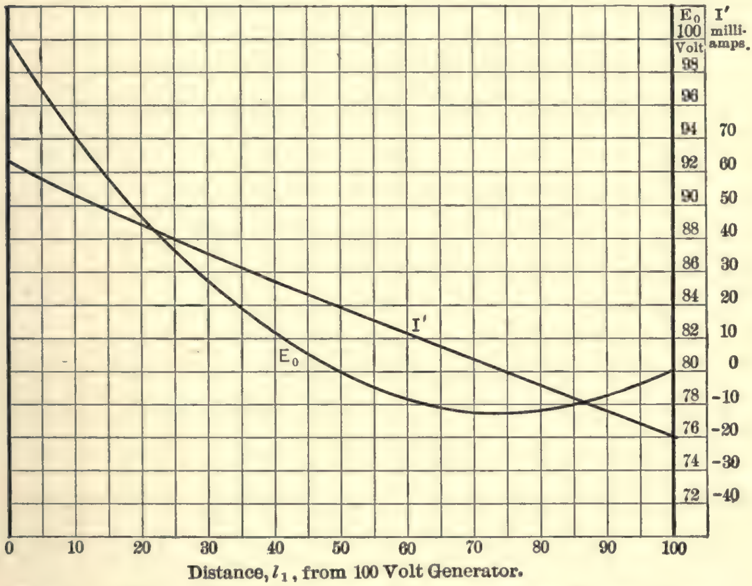

FIG. 8. — Distribution of Voltage and Current along an Unloaded Leaky Line when Supplied with Power at both Ends.
$E_1 = 100$ and $E_2 = 80$ volts.

expressed in terms of the load resistances and the unknown currents. A solution of the simultaneous equations so obtained expresses the unknown currents in terms of the known generator voltages and the load resistances. When applied to a single load fed by a number of transmission systems, this procedure is very simple — leading to as many simultaneous equations as there are transmission systems involved. Figure 7 illustrates a case having two such supply systems. With the possibility of using negative values for the generator voltages in the general solution, the generators may virtually be considered as motors (or loads) of the voltage selected, and the range of applicability of the general solution thereby extended.

Even with three systems of transmission involved in a network, as above mentioned, the expressions both before and after the solution of the simultaneous equations are quite complicated, and with an increase in the number of systems involved or with an addition of intermediate loads the complexity of the mathematical development increases enormously.

CHAPTER III

PERIODIC AND ALTERNATING QUANTITIES. FOURIER'S SERIES AND ANALYSIS OF PERIODIC CURVES

22. General. — The subject of this chapter is one which has been treated very many times. It is discussed here in order that the present volume may be rendered complete and may present to the person engaged in any investigation the means for obtaining with ease and dispatch the analysis of any given curve which is capable of being represented by a Fourier's Series. The matter of analysis by means of a finite number of ordinates, which heretofore has been considerably neglected, is also discussed. As a matter of interest a form of mechanical analyzer which was designed by the writer to facilitate the exposition of the theory of analysis is described.

23. Periodic Quantities and Fourier's Theorem. — A quantity periodic in time may be described as one which passes recurrently, in equal intervals of time, through the same series of values.

An alternating quantity is a special form of periodic quantity, in that in successive equal intervals of time it passes through a series of values which are both positive and negative.

A special form of Fourier's Theorem, which is sufficient for the purpose in hand, states that any continuous, single-valued, finite, and periodic function may be represented by a series of trigonometric sines and cosines — the terms in such a series having frequencies which are multiples of the fundamental frequency of the periodic function.

24. Use of Fourier's Series. — The utility of Fourier's Series in the treatment of alternating current problems is based on the fact that by the separation of a periodic quantity into its constituent terms, a number of simple harmonically varying quantities are obtained which, in further calculation, may each be treated separately by means rigorously applicable only to such simple harmonically varying quantities. Methods of employment of the results of such an analysis will be treated in a later chapter;

this chapter deals only with the methods for separating a given periodic function into the constituent terms.

Usually the given periodic quantity will be specified graphically by a curve, or by a set of numerical data from which such a curve may be plotted. Thus, y may be considered as a function whose value is known for any value of θ.

25. Determination of the Unknown Coefficients by Integration. — Let the scale of the independent variable be so selected that the length of the period is equal to 2π, and for the independent variable to this scale use the quantity or angle θ. Then we assume that

$$\begin{aligned} y = f(\theta) = A_0 + A_1 \cos\theta + A_2 \cos 2\theta + \cdots + A_n \cos n\theta \\ + \cdots + B_1 \sin\theta + B_2 \sin 2\theta + \cdots + B_n \sin n\theta + \cdots \end{aligned} \tag{1}$$

the problem then being to determine the values of the unknown coefficients, A_0, A_1, A_2, etc.; B_1, B_2, B_3, etc.

In the usual manner, the details of which need not be discussed here, we find that by multiplying both sides of equation (1) by $\cos n\theta\, d\theta$ and integrating between the limits 0 and 2π,

$$A_n = \frac{1}{\pi}\int_0^{2\pi} y \cos n\theta\, d\theta, \tag{2}$$

except for the special case of $n = 0$, when the expression

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} y\, d\theta \tag{3}$$

must be used.

By multiplying throughout by $\sin n\theta\, d\theta$ and integrating,

$$B_n = \frac{1}{\pi}\int_0^{2\pi} y \sin n\theta\, d\theta. \tag{4}$$

From equation (2) it is seen that the coefficient A_n is equal to $\frac{1}{\pi}$ times the area under the auxiliary curve

$$y' = y \cos n\theta, \qquad (n \neq 0),$$

when measured between the limits 0 and 2π. Similarly, B_n is equal to $\frac{1}{\pi}$ times area under the auxiliary curve

$$y'' = y \sin n\theta.$$

26. Use of the Auxiliary Curves, $y' = y \cos n\theta$ and $y'' = y \sin n\theta$. — If, from the original curve, these auxiliary curves are calculated and plotted, the areas enclosed could be measured with a planimeter and the values of the various coefficients thereby determined. If a number of equidistant ordinates are known for the original curve, the area of the auxiliary curves may be determined approximately by taking the average height of ordinate of the auxiliary curve as computed (y' or y'') and multiplying by the length of the base, 2π. Dividing the area so obtained by π as in equations (2) and (4) and by 2π as in equation (3), we have the simple result

$$\left.\begin{aligned} A_0 &= (\text{average } y), \\ A_n &= 2\,(\text{avg. } y \cos n\theta), \qquad (n \neq 0), \\ B_n &= 2\,(\text{avg. } y \sin n\theta), \end{aligned}\right\} \tag{5}$$

subject to the errors arising from the approximate manner of obtaining the numerical value of the integral by the use of the average value of a finite number of ordinates.

27. Determination of Coefficients from a Finite Number of Ordinates. — The determination of the various unknown coefficients from a finite number of ordinates will now be considered. The given curve is to be represented by a series of the form of equation (1) in which the number of trigonometric terms may be unlimited.

Divide the wave length, or length of curve which is to be represented by Fourier's Series, into t equal parts, numbering the ordinates 0, 1, 2, 3, . . . $(t-1)$, t. For the fundamental frequency, the total wave length is 2π radians, and therefore ordinate number m has an argument or angle of

$$\frac{2m\pi}{t} \text{ radians.}$$

$$\begin{aligned} y_m = A_0 &+ A_1 \cos \frac{2m\pi}{t} + A_2 \cos 2\frac{2m\pi}{t} + \cdots + A_n \cos n\frac{2m\pi}{t} + \cdots \\ &+ B_1 \sin \frac{2m\pi}{t} + B_2 \sin 2\frac{2m\pi}{t} + \cdots + B_n \sin n\frac{2m\pi}{t} + \cdots. \end{aligned} \tag{6}$$

28. The Cosine Terms. — If we multiply equation (6) throughout by $\cos p\frac{2m\pi}{t}$, where p is any positive integer, and take the summation of all ordinates over a complete wave length,

$$\sum_{m=1}^{m=t} y_m \cos p \frac{2\,m\pi}{t} = A_0 \sum_{m=1}^{m=t} \cos p \frac{2\,m\pi}{t} + A_1 \sum_{m=1}^{m=t} \cos p \frac{2\,m\pi}{t} \cos \frac{2\,m\pi}{t}$$
$$+ \cdots + A_n \sum_{m=1}^{m=t} \cos p \frac{2\,m\pi}{t} \cos n \frac{2\,m\pi}{t} + \cdots$$
$$+ B_1 \sum_{m=1}^{m=t} \cos p \frac{2\,m\pi}{t} \sin \frac{2\,m\pi}{t}$$
$$+ \cdots + B_n \sum_{m=1}^{m=t} \cos p \frac{2\,m\pi}{t} \sin n \frac{2\,m\pi}{t} + \cdots. \quad (7)$$

The summation constituting the coefficient of a general B, or sine, term may be represented by

$$u_n = \sum_{m=1}^{m=t} \cos p \frac{2\,m\pi}{t} \sin n \frac{2\,m\pi}{t},$$

which by the expansion

$$\cos \alpha \sin \beta = \tfrac{1}{2} [\sin (\alpha + \beta) - \sin (\alpha - \beta)]$$

reduces to

$$u_n = \frac{1}{2} \sum_{m=1}^{m=t} \left[\sin \frac{(p+n)\,2\,m\pi}{t} - \sin \frac{(p-n)\,2\,m\pi}{t} \right]. \quad (8)$$

The above summation vanishes for all integral values of p, n, and t, as can readily be seen from the fact that the summation of sines is equivalent to the summation of the projections on the vertical axis of t equal length vectors in a plane spaced under equal angles. Therefore in equation (7) there remain only the cosine, or A, terms.

The summation constituting a general cosine coefficient is

$$z_n = \sum_{m=1}^{m=t} \cos p \frac{2\,m\pi}{t} \cos n \frac{2\,m\pi}{t},$$

which by the expansion

$$\cos \alpha \cos \beta = \tfrac{1}{2} [\cos (\alpha + \beta) + \cos (\alpha - \beta)]$$

reduces to

$$z_n = \frac{1}{2} \sum_{m=1}^{m=t} \left[\cos \frac{(p+n)\,2\,m\pi}{t} + \cos \frac{(p-n)\,2\,m\pi}{t} \right]. \quad (9)$$

In (9) the summation of each term taken separately will vanish unless

$$p + n = it, \qquad \text{or} \qquad p - n = it,$$

that is, unless

$$n = it - p, \qquad \text{or} \qquad n = p - it,$$

in which i represents any integral number. i must assume all values both positive and negative in order that every positive value of n be represented.

Then, since

$$\sum_{m=1}^{m=t} \cos \frac{it\,2\,m\pi}{t} = t, \tag{10}$$

equation (7) reduces finally to

$$\sum_{m=1}^{m=t} y_m \cos p \,\frac{2\,m\pi}{t} = \frac{t}{2} \sum_{i=-\infty}^{i=\infty} \left[A_{(it-p)} + A_{(p-it)} \right], \tag{11}$$

in which any A with a negative subscript is meaningless and therefore to be discarded.

29. The Sine Terms. — Proceeding in a similar manner for the establishment of expressions containing only the B coefficients we multiply equation (6) throughout by $\sin p \,\frac{2\,m\pi}{t}$ and take the summation over a wave length.

$$\begin{aligned}
\sum_{m=1}^{m=t} y_m \sin p \,\frac{2\,m\pi}{t} = {} & A_0 \sum_{m=1}^{m=t} \sin p \,\frac{2\,m\pi}{t} + A_1 \sum_{m=1}^{m=t} \sin p \,\frac{2\,m\pi}{t} \cos \frac{2\,m\pi}{t} \\
& + \cdots + A_n \sum_{m=1}^{m=t} \sin p \,\frac{2\,m\pi}{t} \cos n \,\frac{2\,m\pi}{t} + \cdots \\
& + B_1 \sum_{m=1}^{m=t} \sin p \,\frac{2\,m\pi}{t} \sin \frac{2\,m\pi}{t} \\
& + \cdots + B_n \sum_{m=1}^{m=t} \sin p \,\frac{2\,m\pi}{t} \sin n \,\frac{2\,m\pi}{t} + \cdots .
\end{aligned} \tag{12}$$

In a manner identical with that used for equation (8) it may be shown that the summations constituting the coefficients of the various cosine, or A, terms will all vanish separately when n and p have integral values.

The coefficient of a general sine term is

$$v_n = \sum_{m=1}^{m=t} \sin p \,\frac{2\,m\pi}{t} \sin n \,\frac{2\,m\pi}{t},$$

which by the expansion

$$\sin \alpha \sin \beta = \tfrac{1}{2} [\cos (\alpha - \beta) - \cos (\alpha + \beta)]$$

reduces to

$$v_n = \frac{1}{2} \sum_{m=1}^{m=t} \left[\cos \frac{(p-n)\, 2\, m\pi}{t} - \cos \frac{(p+n)\, 2\, m\pi}{t} \right]. \qquad (13)$$

In equation (13) the separate summations entering v_n are always zero unless

$$p - n = it, \quad \text{or} \quad p + n = it;$$

that is, unless

$$n = p - it, \quad \text{or} \quad n = it - p,$$

where, as before, i must take all negative and positive values in order that all positive values of n be represented.

$$i = \cdots, -3, -2, -1, 0, +1, +2, +3, \cdots$$

Thus, by equation (10),

$$\sum_{m=1}^{m=t} y_m \sin p \frac{2\, m\pi}{t} = \frac{t}{2} \sum_{i=-\infty}^{i=\infty} \left[B_{(p-it)} - B_{(it-p)} \right], \qquad (14)$$

in which, as before, negative subscripts are to be neglected.

30. Interpretation of Results — Cosine Terms. — Equation (11) may be written

$$\sum_{i=-\infty}^{i=\infty} \left[A_{(it-p)} + A_{(p-it)} \right] = \frac{2}{t} \sum_{m=1}^{m=t} y_m \cos p \frac{2\, m\pi}{t}. \qquad (15)$$

Obviously, the right hand member of equation (15) is twice the average value of t ordinates, equally spaced over a wave length, of the auxiliary curve

$$y' = y \cos p\theta,$$

which is exactly the same as the right hand member of the second of equations (5). Consequently, it is apparent that by the use of the finite number of ordinates, t, the numerical value of twice the average ordinate of the auxiliary curve is not equal to the unknown coefficient A_p alone, but to the summation of all the A coefficients of the form indicated by the left hand member of (15) which may be present in the original curve whose analysis is sought. In case an infinite number of ordinates are used, the process of averaging becomes equivalent to an integration, and thus equations (5) and (15) are equivalent for t = infinity.

To be specific, suppose for some particular curve we have measured 18 ordinates over a complete wave length. Also suppose that for the purpose of determining the cosine term of the fifth harmonic we have formed the numerical value of the right hand member of equation (15). Then, $t = 18$ and $p = 5$, so that, neglecting negative subscripts, we have

$$A_5 + A_{13} + A_{23} + A_{31} + A_{41} + A_{49} + A_{59} + \cdots = \frac{2}{18} \sum_{m=1}^{m=18} y_m \cos 5 \frac{2\,m\pi}{18}.$$

Using the expression (5) for this particular case would thus give a correct value for A_5 only in the event of all the A coefficients of the orders A_{13}, A_{23}, A_{31}, A_{41}, etc., being equal to zero, that is, being absent in the particular wave shape under investigation.

31. Interpretation of Results — Sine Terms. — Equation (14) for the sine terms may be similarly treated.

$$\sum_{i=-\infty}^{i=\infty} \left[B_{(p-it)} - B_{(it-p)} \right] = \frac{2}{t} \sum_{m=1}^{m=t} y_m \sin p \frac{2\,m\pi}{t}, \qquad (16)$$

which may be compared with the third of equations (5).

Applying (16) to the numerical example in the preceding paragraph, we note that

$$B_5 - B_{13} + B_{23} - B_{31} + B_{41} - + \cdots = \frac{2}{18} \sum_{m=1}^{m=18} y_m \sin 5 \frac{2\,m\pi}{18}.$$

Similar conditions for the exact determination of B_5 are necessary here as were required for the exact determination of A_5.

32. Particular Orders of Harmonics. — When p is zero or some multiple of $\frac{t}{2}$.

For the purpose of determining A_0 we let $p = 0$, and then the multiplier $\cos p \frac{2\,m\pi}{t}$ in (15) is always unity. The result is then

$$A_0 + A_0 + A_t + A_t + A_{2t} + A_{2t} + \cdots = \frac{2}{t} \sum_{m=1}^{m=t} y_m,$$

or

$$A_0 + A_t + A_{2t} + A_{3t} + \cdots = \frac{1}{t} \sum_{m=1}^{m=t} y_m, \qquad (17)$$

which coincides with the first of (5) except for the disturbing terms, A_t, A_{2t}, A_{3t}, etc.

If p is some multiple of t, the multiplier $\cos p \dfrac{2\,m\pi}{t}$ again reduces to unity and the result of the summation is the same as in (17).

In regard to the sine terms, if p is zero or any multiple of t, it is obvious that the right hand member of equation (16) disappears. Building up the right hand member for p equal to t or any multiple of t, by giving negative and positive values to i, we obtain

$$B_0 - B_0 + B_t - B_t + B_{2t} - B_{2t} + - \cdots = \frac{2}{t}\sum_{m=1}^{m=t} y_m \sin kt \frac{2\,m\pi}{t} = 0, \quad (18)$$

where k = any integer.

If p is an even multiple of $\frac{t}{2}$ it is also a multiple of t, the case just discussed.

For p an odd multiple of $\frac{t}{2}$, equation (15) reduces to

$$A_{\frac{t}{2}} + A_{\frac{3t}{2}} + A_{\frac{5t}{2}} + \cdots = \frac{1}{t}\sum_{m=1}^{m=t} y_m \cos p \frac{2\,m\pi}{t}. \quad (19)$$

This equation reduces to

$$A_{\frac{t}{2}} + A_{\frac{3t}{2}} + A_{\frac{5t}{2}} + \cdots = \frac{1}{t}\sum_{m=1}^{m=t} (-1)^m y_m \quad (19a)$$

which indicates that the determination of any A coefficient whose subscript is an odd multiple of $\frac{t}{2}$ is vitiated by the presence in the given curve of all other coefficients of the same type.

Obviously, the right hand member of (16) is zero when p is an odd multiple of $\frac{t}{2}$. Expanding the left hand member, we have, when p = any odd multiple of $\frac{t}{2}$,

$$B_{\frac{t}{2}} - B_{\frac{t}{2}} + B_{\frac{3t}{2}} - B_{\frac{3t}{2}} + - \cdots = \frac{2}{t}\sum_{m=1}^{m=t} y_m \sin p \frac{2\,m\pi}{t} = 0. \quad (20)$$

33. Limitations to the Use of a Finite Number of Ordinates. — The above development as summarized in equations (15) and (16) is very important in its relation to the actual carrying out of an analysis. It is apparent that if a given curve actually consists of an infinite number of terms, or harmonics, an analysis using only

a finite number of ordinates will not give a correct result. If t ordinates are used, the lowest order harmonic which enters to vitiate the determination of the pth harmonic is that of order $t - p$, as long as p is less than t. Thus, if it is known beforehand that a certain curve contains harmonics up to and including a certain order, to accurately determine any or all of the cosine terms it will be necessary to use a number of ordinates at least twice the order of the highest harmonic known to be present. By equation (20) it is evident that to accurately determine all of the sine terms it will be necessary to use a number of ordinates at least one greater than twice that of the order of the highest harmonic known to be present. This arises from the fact that (20) fails to determine the B coefficient of order $\frac{t}{2}$.

34. Summary. — Although, as was pointed out in the preceding paragraph, it is impossible to secure a rigorously true analysis of a curve containing an infinite number of harmonics by the use of a finite number of ordinates, this fact need cause the engineer, interested primarily in results accurate only within the limits set by the possibilities of physical measurements, much concern. As a rule, in any curve representing physical phenomena the magnitude of the various harmonics present in a periodic curve decreases with increase in order, and if so, it will usually be possible to use a sufficient number of ordinates to cause equations (5) to be sensibly true.

Though there is no intent of discussing here the question of convergence of Fourier's Series or the many very interesting properties of such, it may be well to remark that by the selection of a certain number of ordinates covering one wave length, a similar number of points are established on the original curve through which the curve represented by Fourier's Series resulting from the analysis will pass. This will be true when the subscripts to the A or B terms are taken to a number not to exceed $\frac{t}{2}$, when the ordinates are numbered according to the scheme outlined. Thus by selecting, say, 36 ordinates ($t = 36$), by equations (15) and (16) we may determine

$$A_0, A_1, A_2, \ldots A_{17}, \text{ and } A_{18},$$

as well as

$$B_1, B_2, \ldots B_{16}, \text{ and } B_{17},$$

making in all 36 independent coefficients determined from the 36 ordinates. As pointed out in section 32, equation (20), it is impossible to determine B_{18} in this case. The Fourier's Series thus established will pass exactly through the 36 points selected, so that the only possibility for difference between the original curve and that resulting from the analysis is in the shape of the arch between ordinates. Again, if 35 ordinates equally spaced over a wave length were used, values of

$$A_0, A_1, A_2, \ . \ . \ . \ A_{16}, \text{ and } A_{17},$$
$$B_1, B_2, \ . \ . \ . \ B_{16}, \text{ and } B_{17}$$

would be determined — this time 35 independent coefficients from the 35 ordinates — which would pass a curve exactly through the 35 points. The two sets of coefficients would in general be different, depending upon the magnitudes of the harmonics higher in order than 17 present in the original curve.

As far as engineering and physical sciences are concerned, analysis with a finite number of ordinates can be made sufficient, for in a majority of cases the original curve is specified only by a number of plotted points — the shape of the arch between points being unknown at best. In such cases an analysis as above indicated will give the simplest curve which will represent all of the data. Again, in the analysis of curves which are drawn continuously by some recording device, the accuracy of an analysis will in general be affected more by the inability to secure perfect measurement of instantaneous values than by the failure of the methods given to yield exact results. Illustrative examples of analyses will be shown later.

35. Mechanical Analysis Based upon Integration Processes. — The planimeter is a device which gives directly the numerical value of an integral as typified by the area enclosed by a curve. What a mechanical analyzer must accomplish is to give, by means of a planimeter, the numerical values of integrals of the form indicated in equations (2), (3), and (4). This has been done in several ways, one of which is described here. The apparatus is illustrated schematically in Fig. 9.

In the drawing, $y = f(x)$ is the given curve to be analyzed, and to fix our ideas let us suppose that the coefficient B_n is desired. T is a tracing point with which to follow the given curve. A is the main body carrying the tracing point, which is constrained to a

parallel motion by the mechanism B. A rack, at the end of which is a slide cross-head, is free to move backwards and forwards through the fixed guide G. Thus, the movement of the rack is exactly equal to the x movement of the tracing point T.

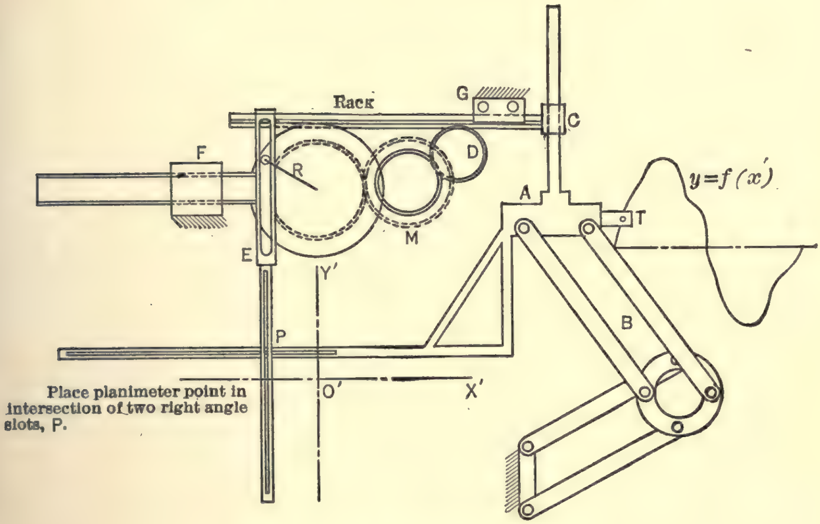

Fig. 9. — An Harmonic Analyzer.

The motion imparted by the rack to the gear D is transmitted through the gear-train M to the face-plate in which is a projecting pin at a distance R from the center as shown. Let the gear-train be so dimensioned that the radius R makes n revolutions to one movement of T from 0 to 2π (one wave length) on the x axis of the given curve. The action of this pin in the slotted cross-head gives a horizontal movement to the bar E dependent upon either the sine or cosine of n times the horizontal displacement of the tracer T.

To the body A is attached the slotted arm H which moves in a vertical direction an amount equal to y on the given curve. Therefore a planimeter point placed in the right-angle intersection of the slots in H and E will have a motion made up of the two components as described.

36. Mathematical Theory of Analyzer. — Take supplementary axes as shown, corresponding to the center of the circle described

by R and to the position occupied by the slot in A when the point T is on the x axis of the given curve. Let the radius R be in its extreme position to the left when T is at the origin of the given curve.

Denote coördinates of the point P with reference to the supplementary axes by x' and y'. Then

$$y' = y,$$
$$x' = -R + (R - R\cos\phi) = -R\cos\phi,$$

where ϕ is the angular displacement of the radius R in a clockwise direction from its initial position to the left. But

$$\phi = nx,$$

and thus

$$\left.\begin{aligned} x' &= -R\cos nx, \\ y' &= y. \end{aligned}\right\} \tag{21}$$

In the new curve whose area may be denoted by S',

$$dS' = y'\,dx' = y\,(Rn\sin nx)\,dx,$$
$$S' = Rn\int_0^{2\pi} y\sin nx.dx. \tag{22}$$

But from equation (4)

$$B_n = \frac{1}{\pi}\int_0^{2\pi} y\sin nx\,dx,$$

so that by comparison with equation (22)

$$B_n = \frac{S'}{\pi Rn}, \tag{23}$$

where S' is the actual area which would be indicated by the planimeter whose tracing point follows the intersection of the two slots in the moving arms.

In a similar manner it can be shown that

$$A_n = \frac{S''}{\pi Rn}, \tag{24}$$

where S'' is the area indicated by the planimeter when the initial position of the radius R is taken in the vertical position above the center.

Obviously, according to equation (4), A_0 would be determined by passing the planimeter over the original curve and dividing the area so obtained by the length of base.

Assuming the apparatus to function perfectly, such a device would yield exact results regardless of the number of harmonics present in the wave. Though of interest, and of value in obtaining a clear idea of the theory of analysis, such a mechanical scheme is not very satisfactory for actual use, in that the labor required to obtain a complete analysis with such a device would be much greater than for procedures based on the measurements of a finite number of ordinates; particularly if tables are available to facilitate the latter form of procedure. Another disadvantage of most mechanical analyzers is that the original curve must be drawn to such a scale that one period or cycle of the fundamental frequency has a certain definite wave length.

37. Tables and Forms for Analysis from a Number of Ordinates. — In regard to analysis by means of a number of ordinates, the various schemes in use are in reality only methods for obtaining rapidly the numerical values of the summations indicated by the right hand members of equations (15) and (16).

Of course it is possible to determine any particular coefficient, for instance A_p, by taking every one of the t ordinates and after multiplying it by the cosine of p times its angle to add it with all similar products from the other ordinates to obtain the summation. This process would have to be repeated for every coefficient desired, and it is apparent that a great amount of time and effort would be consumed, particularly if many ordinates are used. After completing an analysis in such a manner it would be noticed that a great many multiplications involving the same multiplier and multiplicand had been performed. It is the aim of the various tabular schemes in use to minimize the labor of an analysis by eliminating as many as possible of such duplicate operations.

38. Selection of the Number of Ordinates. Separation of Harmonics. — It will in general be advisable to use an even number of ordinates, for then the ordinates of the second half-wave will be located at points on the axis differing by 180 degrees from the abscissæ used in the first half-wave. This permits of the separation of the odd and even harmonics, as will be pointed out.

Since in a majority of cases connected with electrical engineering problems the periodic waves encountered will not contain the harmonics of even order, it is desirable to arrange an analysis scheme in such a way as to make possible the determination of the

odd harmonics only, but at the same time to provide means for a complete analysis, should such be desired.

Since

$$\left.\begin{aligned}\cos p(\beta+\pi) &= \begin{cases} -\cos p\beta \text{ when } p \text{ is odd} \\ \cos p\beta \text{ when } p \text{ is even}\end{cases} \\ \text{and also} \\ \sin p(\beta+\pi) &= \begin{cases} -\sin p\beta \text{ when p is odd} \\ \sin p\beta \text{ when } p \text{ is even}\end{cases}\end{aligned}\right\} \tag{25}$$

it is evident that by taking the average value (one-half the algebraic sum) of two ordinates separated by one-half wave length, or π, we will have the value of that portion of an ordinate which is due to even harmonics only. Similarly, by taking one-half the difference of two ordinates separated by one-half a wave length we obtain the ordinate due to odd harmonics only.

Thus

$$y_m'' = \tfrac{1}{2}\left[y_m + y_{\left(\frac{t}{2}+m\right)}\right] = \text{ordinate due to even harmonics only.} \tag{26}$$

$$y_m' = \tfrac{1}{2}\left[y_m - y_{\left(\frac{t}{2}+m\right)}\right] = \text{ordinate due to odd harmonics only.} \tag{27}$$

This separation of the given data into two sets of data, one containing only even harmonics and the other only odd harmonics, is made possible by the use of an even number of ordinates, t.

39. Description of Tabular Forms for Complete Analysis. — These forms are based on summations of the types (15) and (16), the summation being carried from $m = 0$ to $m = t - 1$, instead of from $m = 1$ to $m = t$. The result is of course the same, but a simplification in notation is obtained which makes the change desirable. The description of the forms follows the order of use in making an analysis from 72 ordinates measured over a complete wave, but as will be apparent, it is not always necessary to use such a large number of ordinates. In fact the forms are arranged so as to accommodate an analysis for both odd and even harmonics from either 72, 36, or 18 ordinates per complete wave, or an analysis for odd harmonics only from either 36 or 18 ordinates measured in a half-wave.

The given data, 72 ordinates per wave, are first separated into two sets of data by equations (26) and (27), one set containing only odd and the other only even harmonics. Form A at the end of the volume is used for this separation.

The 36 values of y' obtained from Form A are then to be used in Form B as original data, and from the sums and differences formed there the odd harmonics are determined and checked by Forms C and D. The process of multiplication by sine or cosine of $p\theta$ is carried out in Form C, while in Form D a number of ordinates are actually computed from the resulting equation, which by coincidence with the original, measured ordinates, establishes the correctness of the numerical computations. Form B is provided with a table for the collecting of the entire results of the analysis.

The 36 values of y'' covering a half wave length of the fundamental frequency still remain, and from these the even harmonics are to be determined. This set of data is equivalent to a set of 36 ordinates covering a complete wave length of the *second harmonic*, which may now be regarded as the fundamental frequency in a new set of harmonics whose frequencies are all multiples of two times the frequency of the original fundamental. Thus the even harmonics, 0, 2, 4, 6, 8, . . . , etc., in the original wave become harmonics of order 0, 1, 2, 3, 4, . . . , etc., in the new system whose fundamental wave length is one-half that of the original.

In the new system, the even harmonics may be separated from the odd in exactly the same manner as was done in Form A for the original system, by combining ordinates separated by one-half of the new wave length. The entire new system is equivalent to a set of 36 ordinates over a complete wave, and the tables are therefore prepared with a notation corresponding to such a measurement of original data, and are described accordingly. It must be remembered that if it is an analysis of a new derived system from 72 ordinates per complete wave which is carried out, the actual frequencies will be twice those indicated by the tabular results — in other words, A_2, A_6, A_{10}, A_{14}, etc., instead of A_1, A_3, A_5, A_7, etc., as indicated in the tabular notation of Forms F and G.

Form E is to be used for separating the set of 36 ordinates per wave length into two sets of 18 ordinates per half wave length, one set containing only the odd and the other only the even harmonics. The 18 values of y' from Form E are to be used in Forms F and G for the determination and checking of the odd harmonics.

If it is known that the original wave contains no even harmonics, Forms F and G will serve immediately for the determination of

the odd harmonics from 18 ordinates measured over a half-wave. Forms B, C, and D may also be used for this purpose with the exception that 36 ordinates per half-wave are used, thus giving a greater accuracy than that obtained with 18 ordinates. After having utilized the 18 values of y' from Form E, the 18 values of y'' still remain, which cover a quarter wave length of the 72 ordinate curve, or a half wave length of the derived 36 ordinate curve, and which are made up from the harmonics of order 4, 8, 12, 16, etc., of the original 72 ordinate curve, or from the harmonics of order 2, 4, 6, 8, etc., of the new 36 ordinate curve.

These eighteen ordinates thus represent a full wave length of a periodic curve whose fundamental frequency is four times that of the original 72 ordinate curve, or twice that of the 36 ordinate curve. Forms H and I are arranged for the analysis of such a full wave length represented by 18 ordinates. Obviously, if the analysis is based upon y'' from Form E, the frequency of the resulting terms must be multiplied by four to be put into the series represented by the original data of Form A or partially expressed by the analysis for odd harmonics in Forms B, C, and D.

Form H as given here is substantially a reproduction of a form or schedule given by Grover in "Bulletin of the Bureau of Standards," Vol. 9, page 646. The reader is referred to this bulletin for a partial bibliography of the subject, and for schedules or forms for analysis to be used with a number of ordinates still smaller than eighteen per complete wave, as covered by Form H.

40. Summary. — To summarize: the forms described provide for a complete analysis for both odd and even harmonics including a constant term, from 72 ordinates per complete wave. To accomplish this end, all of the forms are used — the odd harmonics being determined first, then those whose frequencies are twice those of the odd harmonics, and finally the constant term and those harmonics whose frequencies are four times the odd harmonic frequencies.

In the use of Form I, it is of course to be noticed that twice the values of A_0 and A_9 are obtained, as signified in the notation at the bottom of the blank form for analysis. This comes about as a result of equations (17) and (19), in which the factor $1/t$ is used in place of the factor $2/t$.

If desired, the odd harmonics up to and including the 35th may be determined from a measurement of 36 ordinates over a half

wave as original data, if it is sufficient to represent this half wave only or if it is definitely known that no even harmonics exist in the wave. Also, a complete analysis from 36 ordinates per full wave or an analysis for odd harmonics only from 18 ordinates per half wave is provided for, and finally an analysis for odd and even harmonics from 18 ordinates per full wave.

If in any case the determination of but one particular harmonic is desired, it is of course possible to omit all of the computations

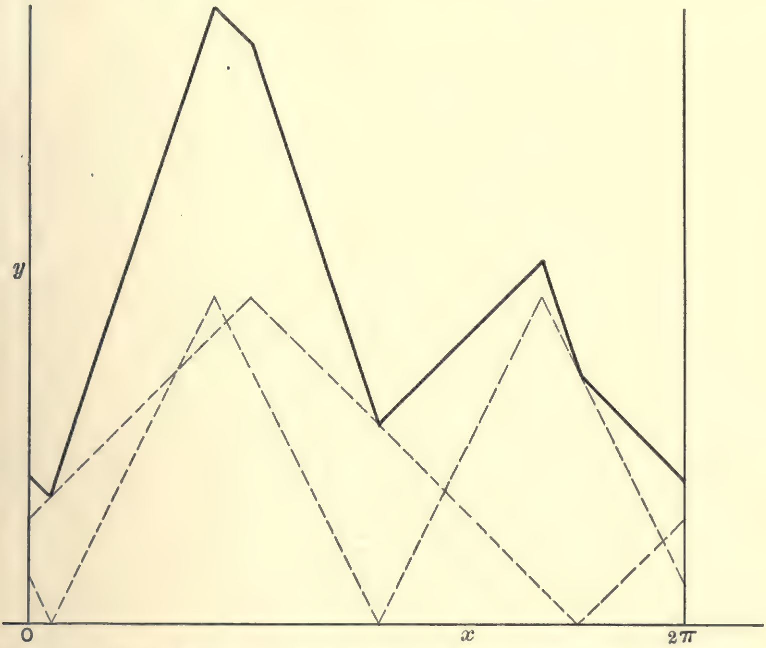

FIG. 10. — The Broken-line Periodic Curve Represented by Equation (28).

except those directly involved in the various steps leading thereto. A little experience in the use of the forms will immediately suggest simplifications of procedure which may be possible in special cases.

Analysis forms of the type shown here have been in constant use by the writer, and in all respects have been found the most satisfactory means for separating a periodic curve into its constituent harmonics.

41. Numerical Illustration as Check on Tabular Forms. — In order to check the analysis forms as well as to furnish an illustration

of what might be expected in the way of accuracy from their use, an analysis of a curve whose constituent harmonics could be determined by integration was made. The curve used is made up of a number of straight lines whose equations are known so that the numerical values of the ordinates may be computed with accuracy. The curve is made up of the sum of two curves, one containing odd and the other even harmonics, so that both odd and even harmonics are present in the resultant. Fig. 10 shows the wave shape resulting from the sum of the two component curves. Its equation, when x is measured in radians, is

$$y = y' + y'' = \frac{\pi}{2} -$$
$$\frac{4}{\pi}\left[\cos(x+1) + \frac{1}{9}\cos 3(x+1) + \frac{1}{25}\cos 5(x+1) + \cdots\right] + \frac{\pi}{2} -$$
$$\frac{4}{\pi}\left[\cos 2(x-0.2) + \frac{1}{9}\cos 6(x-0.2) + \frac{1}{25}\cos 10(x-0.2) + \cdots\right]. \quad (28)$$

Expanding $\cos(x + \alpha)$ by $\cos\alpha\cos x - \sin\alpha\sin x$, the A and B coefficients are obtained, and the complete curve, $y = f(x)$, is represented by a Fourier's Series in which the coefficients have the values given in Table V.

From the known line slopes and intercepts the 72 ordinate values as given in Table VI were obtained. These ordinates were equally spaced over a wave length.

TABLE V

COEFFICIENTS OF FOURIER'S SERIES REPRESENTING THE BROKEN LINE CURVE. (EQUATION 28.)

Odd harmonics			Even harmonics		
n	A_n	B_n	n	A_n	B_n
0	....	....	0	+3.141593	....
1	−0.68793	+1.07140	2	−1.17273	−0.49581
3	+0.14005	+0.019966	6	−0.051264	−0.131859
5	−0.014447	−0.048837	10	+0.021194	−0.046310
7	−0.019590	+0.017071	14	+0.025622	−0.0043209
9	+0.014322	+0.0064781	18	+0.014096	+0.0069560
11	−0.00004657	−0.010522	22	+0.0032343	+0.0100132
13	−0.0068366	+0.0031655	26	−0.0035298	+0.0066560
15	+0.0042990	+0.0036798	30	−0.0054335	+0.0015811
17	+0.0012120	−0.0042356	34	−0.0038303	−0.0021768
19	−0.0034871	+0.00052861	38	−0.00088618	−0.0034138
21	+0.0015814	+0.0024155	42	+0.0014993	−0.0024673
23	+0.0012824	−0.0020367	46	+0.0023463	−0.00053648
25	−0.0020192	−0.00026962	50	+0.0017093	+0.0011082
27	+0.00051023	+0.0016703	54	+0.00033942	+0.0017132
29	+0.0011325	−0.0010047	58	−0.00086036	+0.0012457
31	−0.0012120	−0.00053532	62	−0.0013066	+0.00021941
33	+0.00001552	+0.0011691	66	−0.00094224	−0.00069223
35	+0.00093928	−0.00044503	70	−0.00014212	−0.0010296
37	−0.00071187	−0.00059852	74	+0.00057232	−0.00073312
39	−0.00022322	+0.00080679	78	+0.00083222	−0.00009020
41	+0.00074782	−0.00012015	82	+0.00058318	+0.00048332
43	−0.00038225	−0.00057277	86	+0.00005418	+0.00068647
45	−0.00033030	+0.00053501	90	−0.00041517	+0.00047218
47	+0.00057196	+0.00007122	94	−0.00057568	−0.00002855
49	−0.00015940	−0.00050577	98	−0.00038785	−0.00036164
51	−0.00036330	+0.00032809	102	−0.00000996	−0.00048942
53	+0.00041626	+0.00017947	106	+0.00031867	−0.00032234
55	−0.00000931	−0.00042079	110	+0.00042088	+0.00000373
	etc.	etc.		etc.	etc.

TABLE VI

SEVENTY-TWO ORDINATES FOR CHECK ANALYSIS OF PERIODIC CURVE

m	Y_m	m	Y_m	m	Y_m	m	Y_m
0	1.40000	18	5.31239	36	2.54159	54	3.31239
1	1.31274	19	5.57419	37	2.27980	55	3.39966
2	1.22546	20	5.83599	38	2.01799	56	3.48692
3	1.38540	21	5.85059	39	2.00339	57	3.32699
4	1.64720	22	5.76332	40	2.09066	58	3.06519
5	1.90899	23	5.67606	41	2.17792	59	2.80339
6	2.17080	24	5.58880	42	2.26519	60	2.54160
7	2.43260	25	5.42138	43	2.35246	61	2.35993
8	2.69439	26	5.15959	44	2.43972	62	2.27267
9	2.95620	27	4.89779	45	2.52699	63	2.18540
10	3.21799	28	4.63598	46	2.61426	64	2.09813
11	3.47979	29	4.37419	47	2.70152	65	2.01086
12	3.74160	30	4.11239	48	2.78880	66	1.92360
13	4.00339	31	3.85058	49	2.87606	67	1.83633
14	4.26519	32	3.58879	50	2.96332	68	1.74906
15	4.52699	33	3.32699	51	3.05059	69	1.66180
16	4.78879	34	3.06520	52	3.13786	70	1.57454
17	5.05059	35	2.80339	53	3.22512	71	1.48726

The above table contains the ordinates for each five degrees along the axis of x. These ordinates represent the values to which the infinite series (28) would converge for the assumed abscissæ.

From the data given in Table VI a complete analysis was made, using all of the forms. The values of coefficients derived thereby are given in Table VII.

The results of the complete analysis as tabulated in Table VII are to be compared with the corresponding values in Table V, which contains the true values of the coefficients. The differences noted arise from the effect of the disturbing terms in the summation-equations (15) and (16). For example: the true value of A_2 is -1.17273, while from the analysis we obtain $A_2 = -1.17205$, in this case rather close agreement, but still not exact. By equation (15), however, the value -1.17205 should be equal to the sum of the true values of

$$A_2 + A_{70} + A_{74} + A_{142} + A_{146} + \cdots.$$

Forming this sum partially from Table V,

$$A_2 + A_{70} + A_{74} = -1.17230,$$

a value nearer the result of analysis than the actual value of A_2, but still differing therefrom on account of the neglected terms in the infinite series,

$$A_{142} + A_{146} + A_{214} + A_{218} + \cdots .$$

Again, by equation (16), using the true values from Table V,

$$B_2 - B_{70} + B_{74} = -0.49551,$$

a quantity very nearly equal to -0.49543, which is the value of B_2 resulting from the analysis.

TABLE VII

COEFFICIENTS OF FOURIER'S SERIES DERIVED BY COMPLETE ANALYSIS FROM 72 ORDINATES PER COMPLETE WAVE, FIG. 10

n	A_n	B_n	n	A_n	B_n
0	....	....	0	3.14159	....
1	−0.68772	1.07106	2	−1.17205	−0.49543
3	0.13966	0.01992	4	0	0
5	−0.01432	−0.04846	6	−0.05125	−0.13102
7	−0.01930	0.01679	8	0	0
9	0.01393	0.00633	10	0.02040	−0.04578
11	−0.00001	−0.01009	12	0	0
13	−0.00645	0.00294	14	0.02344	−0.00912
15	0.00392	0.00342	16	0	0
17	0.00113	−0.00376	18	0.01373	0.00566
19	−0.00300	0.00039	20	0	0
21	0.00123	0.00201	22	0.00419	0.00869
23	0.00105	−0.00152	24	0	0
25	−0.00141	−0.00028	26	−0.00153	0.00651
27	0.00022	0.00109	28	0	0
29	0.00069	−0.00045	30	−0.00368	0.00334
31	−0.00046	−0.00036	32	0	0
33	−0.00016	0.00035	34	−0.00403	0.00094
35	0.00020	0.00011	36	0	....

An inspection of the values of the small coefficients of the high frequency terms will reveal a very considerable error in the analysis. These errors arise from the presence in the original curve of an infinite number of harmonics. In the analysis of any experimentally obtained curve such a condition would not be expected to occur, and therefore the results would not be so much in error. In all events, the 72 points selected on the original curve are represented by the resulting equation.

42. Comparative Accuracy. — To illustrate the deviation of the curve obtained by analysis, from the original curve, an analysis

of the preceding broken line curve was made from 18 ordinates (that is, every fourth ordinate as tabulated in Table VI was used in Forms H and I as original data) with the results shown in Table VIII.

TABLE VIII

COEFFICIENTS OF FOURIER'S SERIES DERIVED BY COMPLETE ANALYSIS FROM 18 ORDINATES PER COMPLETE WAVE, FIG. 10

n	A_n	B_n	n	A_n	B_n
0	3.15533	....	5	−0.02018	−0.05255
1	−0.68939	1.07510	6	−0.05493	−0.13436
2	−1.17608	−0.49637	7	−0.02003	0.02705
3	0.14465	0.01817	8	0.01886	0.05229
4	0.02762	0.01780	9	0.01415	....

Again, for illustration, by equation (16), using true values of coefficients from Table V, and for $t = 18$ and $p = 6$,

$$B_6 - B_{12} + B_{24} - B_{30} + B_{42} - B_{48} + B_{60} - B_{66} + B_{78} - B_{84} + B_{96} - B_{102} = -0.13482,$$

which is nearly equal to the value −0.13436 as given by the 18 ordinate analysis for the coefficient B_6.

Figure 11 shows the original broken line curve and also the curve represented by the finite Fourier's Series having the coefficients given in Table VIII. The eighteen ordinates from which the analysis was made are shown in the figure, and corresponding ordinates in both curves should theoretically be equal. The lower, smooth curve, represented by the finite series containing terms up to the ninth harmonic was drawn by means of a curve tracing and analyzing machine designed and constructed by Professor J. N. LeConte, of the University of California. Small differences in the corresponding ordinates of the two curves may be attributed to slight errors in adjustment of the curve tracing apparatus. The approximation of the derived curve to the original is clearly shown, nevertheless.

In order to facilitate the computation of ordinate values and the plotting of wave shapes from Fourier's Series with given coefficients by those who are not fortunate enough to have a curve-tracing device at hand, Table VIII A has been prepared, from which

the trigonometric functions of multiple angles (up to and including the 36th harmonic) may be obtained by finding the function of some angle included in the first quadrant. The upper algebraic sign is for cosine and the lower algebraic sign for sine functions.

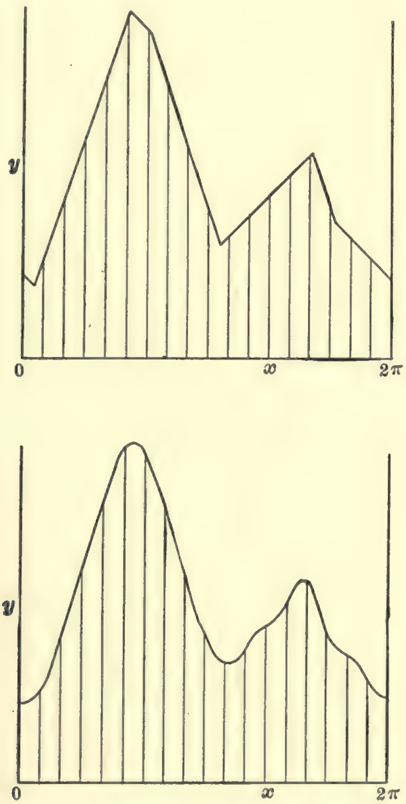

FIG. 11. — The Original Curve and the Curve Represented by the Results of an Eighteen Ordinate Analysis.

TABLE VIII A

Table for Trigonometric Functions of Multiple Angles

$\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} n\theta = \left\{\begin{matrix}\pm\cos\\ \pm\sin\end{matrix}\right\} \phi$

Values of ϕ

Example

$\cos 23 \times 75° = \cos 75°$
$\sin 23 \times 75° = -\sin 75°$

θ \ n	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0
5	5	10	15	20	25	30	35	40	45
10	10	20	30	40	50	60	70	80	90
15	15	30	45	60	75	90	∓75	∓60	∓45
20	20	40	60	80	∓80	∓60	∓40	∓20	− 0
25	25	50	75	∓80	∓55	∓30	∓ 5	$^{-}_{-}20$	$^{-}_{-}45$
30	30	60	90	∓60	∓30	− 0	$^{-}_{-}30$	$^{-}_{-}60$	−90
35	35	70	∓75	∓40	∓ 5	$^{-}_{-}30$	$^{-}_{-}65$	±80	±45
40	40	80	∓60	∓20	$^{-}_{-}20$	$^{-}_{-}60$	±80	±40	+ 0
45	45	90	∓45	− 0	$^{-}_{-}45$	−90	±45	+ 0	$^{+}_{+}45$
50	50	∓80	∓30	$^{-}_{-}20$	$^{-}_{-}70$	±60	±10	$^{+}_{+}40$	+90
55	55	∓70	∓15	$^{-}_{-}40$	±85	±30	$^{+}_{+}25$	$^{+}_{+}80$	∓45
60	60	∓60	− 0	$^{-}_{-}60$	±60	+ 0	$^{+}_{+}60$	∓60	− 0
65	65	∓50	$^{-}_{-}15$	$^{-}_{-}80$	±35	$^{+}_{+}30$	∓85	∓20	$^{-}_{-}45$
70	70	∓40	$^{-}_{-}30$	±80	±10	$^{+}_{+}60$	∓50	$^{-}_{-}20$	−90
75	75	∓30	$^{-}_{-}45$	±60	$^{+}_{+}15$	+90	∓15	$^{-}_{-}60$	±45
80	80	∓20	$^{-}_{-}60$	±40	$^{+}_{+}40$	∓60	$^{-}_{-}20$	±80	+ 0
85	85	∓10	$^{-}_{-}75$	±20	$^{+}_{+}65$	∓30	$^{-}_{-}55$	±40	$^{+}_{+}45$
90	90	− 0	−90	+ 0	+90	− 0	−90	+ 0	+90
95	∓85	$^{-}_{-}10$	±75	$^{+}_{+}20$	∓65	$^{-}_{-}30$	±55	$^{+}_{+}40$	∓45
100	∓80	$^{-}_{-}20$	±60	$^{+}_{+}40$	∓40	$^{-}_{-}60$	±20	$^{+}_{+}80$	− 0
105	∓75	$^{-}_{-}30$	±45	$^{+}_{+}60$	∓15	−90	$^{+}_{+}15$	∓60	$^{-}_{-}45$
110	∓70	$^{-}_{-}40$	±30	$^{+}_{+}80$	$^{-}_{-}10$	±60	$^{+}_{+}50$	∓20	−90
115	∓65	$^{-}_{-}50$	±15	∓80	$^{-}_{-}35$	±30	$^{+}_{+}85$	$^{-}_{-}20$	±45
120	∓60	$^{-}_{-}60$	+ 0	∓60	$^{-}_{-}60$	+ 0	∓60	$^{-}_{-}60$	+ 0
125	∓55	$^{-}_{-}70$	$^{+}_{+}15$	∓40	$^{-}_{-}85$	$^{+}_{+}30$	∓25	±80	$^{+}_{+}45$
130	∓50	$^{-}_{-}80$	$^{+}_{+}30$	∓20	±70	$^{+}_{+}60$	$^{-}_{-}10$	±40	+90
135	∓45	−90	$^{+}_{+}45$	− 0	±45	+90	$^{-}_{-}45$	+ 0	∓45
140	∓40	±80	$^{+}_{+}60$	$^{-}_{-}20$	±20	∓60	$^{-}_{-}80$	$^{+}_{+}40$	− 0
145	∓35	±70	$^{+}_{+}75$	$^{-}_{-}40$	$^{+}_{+}5$	∓30	±65	$^{+}_{+}80$	$^{-}_{-}45$
150	∓30	±60	+90	$^{-}_{-}60$	$^{+}_{+}30$	− 0	±30	∓60	−90
155	∓25	±50	∓75	$^{-}_{-}80$	$^{+}_{+}55$	$^{-}_{-}30$	$^{+}_{+}5$	∓20	±45
160	∓20	±40	∓60	±80	$^{+}_{+}80$	$^{-}_{-}60$	$^{+}_{+}40$	$^{-}_{-}20$	+ 0
165	∓15	±30	∓45	±60	∓75	−90	$^{+}_{+}75$	$^{-}_{-}60$	$^{+}_{+}45$
170	∓10	±20	∓30	±40	∓50	±60	∓70	±80	+90
175	∓ 5	±10	∓15	±20	∓25	±30	∓35	±40	∓45
180	− 0	+ 0	− 0	+ 0	− 0	+ 0	− 0	+ 0	− 0

TABLE VIII A — (*Continued*)

Table for Trigonometric Functions of Multiple Angles

θ \ n	10	11	12	13	14	15	16	17	18
0	0	0	0	0	0	0	0	0	0
5	50	55	60	65	70	75	80	85	90
10	∓ 80	∓ 70	∓ 60	∓ 50	∓ 40	∓ 30	∓ 20	∓ 10	-0
15	∓ 30	∓ 15	-0	$^{-}_{-}15$	$^{-}_{-}30$	$^{-}_{-}45$	$^{-}_{-}60$	$^{-}_{-}75$	-90
20	$^{-}_{-}20$	$^{-}_{-}40$	$^{-}_{-}60$	$^{-}_{-}80$	± 80	± 60	± 40	± 20	$+0$
25	$^{-}_{-}70$	± 85	± 60	± 35	± 10	$^{+}_{+}15$	$^{+}_{+}40$	$^{+}_{+}65$	$+90$
30	± 60	± 30	$+0$	$^{+}_{+}30$	$^{+}_{+}60$	$+90$	∓ 60	∓ 30	-0
35	± 10	$^{+}_{+}25$	$^{+}_{+}60$	∓ 85	∓ 50	∓ 15	$^{-}_{-}20$	$^{-}_{-}55$	-90
40	$^{+}_{+}40$	$^{+}_{+}80$	∓ 60	∓ 20	$^{-}_{-}20$	$^{-}_{-}60$	± 80	± 40	$+0$
45	$+90$	∓ 45	-0	$^{-}_{-}45$	-90	± 45	$+0$	$^{+}_{+}45$	$+90$
50	∓ 40	$^{-}_{-}10$	$^{-}_{-}60$	± 70	± 20	$^{+}_{+}30$	$^{+}_{+}80$	∓ 50	-0
55	$^{-}_{-}10$	$^{-}_{-}65$	± 60	± 5	$^{+}_{+}50$	∓ 75	∓ 20	$^{-}_{-}35$	-90
60	$^{-}_{-}60$	± 60	$+0$	$^{+}_{+}60$	∓ 60	-0	$^{-}_{-}60$	± 60	$+0$
65	± 70	± 5	$^{+}_{+}60$	∓ 55	$^{-}_{-}10$	$^{-}_{-}75$	± 40	$^{+}_{+}25$	$+90$
70	± 20	$^{+}_{+}50$	∓ 60	$^{-}_{-}10$	$^{-}_{-}80$	± 30	$^{+}_{+}40$	∓ 70	-0
75	$^{+}_{+}30$	∓ 75	-0	$^{-}_{-}75$	± 30	$^{+}_{+}45$	∓ 60	$^{-}_{-}15$	-90
80	$^{+}_{+}80$	∓ 20	$^{-}_{-}60$	± 40	$^{+}_{+}40$	∓ 60	$^{-}_{-}20$	± 80	$+0$
85	∓ 50	$^{-}_{-}35$	± 60	$^{+}_{+}25$	∓ 70	$^{-}_{-}15$	± 80	$^{+}_{+}5$	$+90$
90	-0	-90	$+0$	$+90$	-0	-90	$+0$	$+90$	-0
95	$^{-}_{-}50$	± 35	$^{+}_{+}60$	∓ 25	$^{-}_{-}70$	± 15	$^{+}_{+}80$	∓ 5	-90
100	± 80	$^{+}_{+}20$	∓ 60	$^{-}_{-}40$	± 40	$^{+}_{+}60$	∓ 20	$^{-}_{-}80$	$+0$
105	± 30	$^{+}_{+}75$	-0	± 75	$^{+}_{+}30$	∓ 45	$^{-}_{-}60$	± 15	$+90$
110	$^{+}_{+}20$	∓ 50	$^{-}_{-}60$	± 10	∓ 80	$^{-}_{-}30$	± 40	$^{+}_{+}70$	-0
115	$^{+}_{+}70$	$^{-}_{-}5$	± 60	$^{+}_{+}55$	∓ 10	± 75	$^{+}_{+}40$	∓ 25	-90
120	∓ 60	$^{-}_{-}60$	$+0$	∓ 60	$^{-}_{-}60$	$+0$	∓ 60	$^{-}_{-}60$	$+0$
125	∓ 10	± 65	$^{+}_{+}60$	$^{-}_{-}5$	± 50	$^{+}_{+}75$	$^{-}_{-}20$	± 35	$+90$
130	$^{-}_{-}40$	± 10	∓ 60	$^{-}_{-}70$	$^{+}_{+}20$	∓ 30	± 80	$^{+}_{+}50$	-0
135	-90	$^{+}_{+}45$	-0	± 45	$+90$	$^{-}_{-}45$	$+0$	∓ 45	-90
140	± 40	∓ 80	$^{-}_{-}60$	$^{+}_{+}20$	∓ 20	± 60	$^{+}_{+}80$	$^{-}_{-}40$	$+0$
145	$^{+}_{+}10$	∓ 25	± 60	$^{+}_{+}85$	$^{-}_{-}50$	$^{+}_{+}15$	∓ 20	± 55	$+90$
150	$^{+}_{+}60$	$^{-}_{-}30$	$+0$	∓ 30	± 60	$+90$	$^{-}_{-}60$	$^{+}_{+}30$	-0
155	∓ 70	$^{-}_{-}85$	$^{+}_{+}60$	$^{-}_{-}35$	$^{+}_{+}10$	∓ 15	± 40	∓ 65	-90
160	∓ 20	± 40	∓ 60	± 80	$^{+}_{+}80$	$^{-}_{-}60$	$^{+}_{+}40$	$^{-}_{-}20$	$+0$
165	$^{-}_{-}30$	$^{+}_{+}15$	-0	± 15	∓ 30	± 45	∓ 60	± 75	$+90$
170	$^{-}_{-}80$	$^{+}_{+}70$	$^{-}_{-}60$	$^{+}_{+}50$	$^{-}_{-}40$	$^{+}_{+}30$	$^{-}_{-}20$	$^{+}_{+}10$	-0
175	± 50	∓ 55	± 60	∓ 65	± 70	∓ 75	± 80	∓ 85	-90
180	$+0$	-0	$+0$	-0	$+0$	-0	$+0$	-0	$+0$

TABLE VIII A—(*Continued*)

Table for Trigonometric Functions of Multiple Angles

θ \ n	19	20	21	22	23	24	25	26	27
0	0	0	0	0	0	0	0	0	0
5	∓85	∓80	∓75	∓70	∓65	∓60	∓55	∓50	∓45
10	$^{-}_{-}10$	$^{-}_{-}20$	$^{-}_{-}30$	$^{-}_{-}40$	$^{-}_{-}50$	$^{-}_{-}60$	$^{-}_{-}70$	$^{-}_{-}80$	−90
15	±75	±60	±45	±30	±15	+ 0	$^{+}_{+}15$	$^{+}_{+}30$	$^{+}_{+}45$
20	$^{+}_{+}20$	$^{+}_{+}40$	$^{+}_{+}60$	$^{+}_{+}80$	∓80	∓60	∓40	∓20	− 0
25	∓65	∓40	∓15	$^{-}_{-}10$	$^{-}_{-}35$	$^{-}_{-}60$	$^{-}_{-}85$	±70	±45
30	$^{-}_{-}30$	$^{-}_{-}60$	−90	±60	±30	+ 0	$^{+}_{+}30$	$^{+}_{+}60$	+90
35	±55	±20	$^{+}_{+}15$	$^{+}_{+}50$	$^{+}_{+}85$	∓60	∓25	$^{-}_{-}10$	$^{-}_{-}45$
40	$^{+}_{+}40$	$^{+}_{+}80$	∓60	∓20	$^{-}_{-}20$	$^{-}_{-}60$	±80	±40	+ 0
45	∓45	− 0	$^{-}_{-}45$	−90	±45	+ 0	$^{+}_{+}45$	+90	∓45
50	$^{-}_{-}50$	±80	±30	$^{+}_{+}20$	$^{+}_{+}70$	∓60	∓10	$^{-}_{-}40$	−90
55	±35	$^{+}_{+}20$	$^{+}_{+}75$	∓50	$^{-}_{-}5$	$^{-}_{-}60$	±65	±10	$^{+}_{+}45$
60	$^{+}_{+}60$	∓60	− 0	$^{-}_{-}60$	±60	+ 0	$^{+}_{+}60$	∓60	− 0
65	∓25	$^{-}_{-}40$	±75	±10	$^{+}_{+}55$	∓60	$^{-}_{-}5$	$^{-}_{-}70$	±45
70	$^{-}_{-}70$	±40	$^{+}_{+}30$	∓80	∓10	$^{-}_{-}60$	±50	$^{+}_{+}20$	+90
75	±15	$^{+}_{+}60$	∓45	$^{-}_{-}30$	±75	+ 0	$^{+}_{+}75$	∓30	$^{-}_{-}45$
80	$^{+}_{+}80$	∓20	$^{-}_{-}60$	±40	$^{+}_{+}40$	∓60	$^{-}_{-}20$	±80	+ 0
85	∓ 5	$^{-}_{-}80$	±15	$^{+}_{+}70$	∓25	$^{-}_{-}60$	±35	$^{+}_{+}50$	∓45
90	−90	+ 0	+90	− 0	−90	+ 0	+90	− 0	−90
95	$^{+}_{+}5$	∓80	$^{-}_{-}15$	±70	$^{+}_{+}25$	∓60	$^{-}_{-}35$	±50	$^{+}_{+}45$
100	∓80	$^{-}_{-}20$	±60	$^{+}_{+}40$	∓40	$^{-}_{-}60$	±20	$^{+}_{+}80$	− 0
105	$^{-}_{-}15$	±60	$^{+}_{+}45$	∓30	$^{-}_{-}75$	+ 0	∓75	$^{-}_{-}30$	±45
110	±70	$^{+}_{+}40$	∓30	$^{-}_{-}80$	$^{+}_{+}10$	∓60	$^{-}_{-}50$	±20	+90
115	$^{+}_{+}25$	∓40	$^{-}_{-}75$	$^{+}_{+}10$	∓55	$^{-}_{-}60$	± 5	∓70	$^{-}_{-}45$
120	∓60	$^{-}_{-}60$	+ 0	∓60	$^{-}_{-}60$	+ 0	∓60	$^{-}_{-}60$	+ 0
125	$^{-}_{-}35$	±20	∓75	$^{-}_{-}50$	± 5	∓60	$^{-}_{-}65$	$^{+}_{+}10$	∓45
130	±50	$^{+}_{+}80$	$^{-}_{-}30$	±20	∓70	$^{-}_{-}60$	$^{+}_{+}10$	∓40	−90
135	$^{+}_{+}45$	− 0	±45	+90	$^{-}_{-}45$	+ 0	∓45	−90	$^{+}_{+}45$
140	∓40	±80	$^{+}_{+}60$	$^{-}_{-}20$	±20	∓60	$^{-}_{-}80$	$^{+}_{+}40$	− 0
145	$^{-}_{-}55$	$^{+}_{+}20$	∓15	±50	∓85	$^{-}_{-}60$	$^{+}_{+}25$	∓10	±45
150	±30	∓60	−90	$^{+}_{+}60$	$^{-}_{-}30$	+ 0	∓30	±60	+90
155	$^{+}_{+}65$	$^{-}_{-}40$	$^{+}_{+}15$	∓10	±35	∓60	±85	$^{+}_{+}70$	$^{-}_{-}45$
160	∓20	±40	∓60	±80	$^{+}_{+}80$	$^{-}_{-}60$	$^{+}_{+}40$	$^{-}_{-}20$	+ 0
165	$^{-}_{-}75$	$^{+}_{+}60$	$^{-}_{-}45$	$^{+}_{+}30$	$^{-}_{-}15$	+ 0	∓15	±30	∓45
170	±10	∓20	±30	∓40	±50	∓60	±70	∓80	−90
175	$^{+}_{+}85$	$^{-}_{-}80$	$^{+}_{+}75$	$^{-}_{-}70$	$^{+}_{+}65$	$^{-}_{-}60$	$^{+}_{+}55$	$^{-}_{-}50$	$^{+}_{+}45$
180	− 0	+ 0	− 0	+ 0	− 0	+ 0	− 0	+ 0	− 0

TABLE VIII A — (*Concluded*)

Table for Trigonometric Functions of Multiple Angles

θ \ n	28	29	30	31	32	33	34	35	36
0	0	0	0	0	0	0	0	0	0
5	∓ 40	∓ 35	∓ 30	∓ 25	∓ 20	∓ 15	∓ 10	∓ 5	-0
10	± 80	± 70	± 60	± 50	± 40	± 30	± 20	± 10	$+0$
15	$\substack{+\\+}60$	$\substack{+\\+}75$	$+90$	∓ 75	∓ 60	∓ 45	∓ 30	∓ 15	-0
20	$\substack{-\\-}20$	$\substack{-\\-}40$	$\substack{-\\-}60$	$\substack{-\\-}80$	± 80	± 60	± 40	± 20	$+0$
25	± 20	$\substack{+\\+}5$	$\substack{+\\+}30$	$\substack{+\\+}55$	$\substack{+\\+}80$	∓ 75	∓ 50	∓ 25	-0
30	∓ 60	∓ 30	-0	$\substack{-\\-}30$	$\substack{-\\-}60$	-90	± 60	± 30	$+0$
35	$\substack{-\\-}80$	± 65	± 30	$\substack{+\\+}5$	$\substack{+\\+}40$	$\substack{+\\+}75$	∓ 70	∓ 35	-0
40	$\substack{+\\+}40$	$\substack{+\\+}80$	∓ 60	∓ 20	$\substack{-\\-}20$	$\substack{-\\-}60$	± 80	± 40	$+0$
45	-0	$\substack{-\\-}45$	-90	± 45	$+0$	$\substack{+\\+}45$	$+90$	∓ 45	-0
50	± 40	$\substack{+\\+}10$	$\substack{+\\+}60$	∓ 70	∓ 20	$\substack{-\\-}30$	$\substack{-\\-}80$	± 50	$+0$
55	∓ 80	∓ 25	$\substack{-\\-}30$	$\substack{-\\-}85$	± 40	$\substack{+\\+}15$	$\substack{+\\+}70$	∓ 55	-0
60	$\substack{-\\-}60$	± 60	$+0$	$\substack{+\\+}60$	∓ 60	-0	$\substack{-\\-}60$	± 60	$+0$
65	$\substack{+\\+}20$	$\substack{+\\+}85$	∓ 30	$\substack{-\\-}35$	± 80	± 15	$\substack{+\\+}50$	∓ 65	-0
70	∓ 20	$\substack{-\\-}50$	± 60	$\substack{+\\+}10$	$\substack{+\\+}80$	∓ 30	$\substack{-\\-}40$	± 70	$+0$
75	± 60	$\substack{+\\+}15$	$+90$	∓ 15	$\substack{-\\-}60$	± 45	$\substack{+\\+}30$	∓ 75	-0
80	$\substack{+\\+}80$	∓ 20	$\substack{-\\-}60$	± 40	$\substack{+\\+}40$	∓ 60	$\substack{-\\-}20$	± 80	$+0$
85	$\substack{-\\-}40$	± 55	$\substack{+\\+}30$	∓ 65	$\substack{-\\-}20$	± 75	$\substack{+\\+}10$	∓ 85	-0
90	$+0$	$+90$	-0	-90	$+0$	$+90$	-0	-90	$+0$
95	∓ 40	$\substack{-\\-}55$	± 30	$\substack{+\\+}65$	∓ 20	$\substack{-\\-}75$	± 10	$\substack{+\\+}85$	-0
100	± 80	$\substack{+\\+}20$	∓ 60	$\substack{-\\-}40$	± 40	$\substack{+\\+}60$	∓ 20	$\substack{-\\-}80$	$+0$
105	$\substack{+\\+}60$	∓ 15	-90	$\substack{+\\+}15$	∓ 60	$\substack{-\\-}45$	± 30	$\substack{+\\+}75$	-0
110	$\substack{-\\-}20$	± 50	$\substack{+\\+}60$	∓ 10	± 80	$\substack{+\\+}30$	∓ 40	$\substack{-\\-}70$	$+0$
115	± 20	∓ 85	$\substack{-\\-}30$	± 35	$\substack{+\\+}80$	$\substack{-\\-}15$	± 50	$\substack{+\\+}65$	-0
120	∓ 60	$\substack{-\\-}60$	$+0$	∓ 60	$\substack{-\\-}60$	$+0$	∓ 60	$\substack{-\\-}60$	$+0$
125	$\substack{-\\-}80$	$\substack{+\\+}25$	∓ 30	± 85	$\substack{+\\+}40$	∓ 15	± 70	$\substack{+\\+}55$	-0
130	$\substack{+\\+}40$	∓ 10	± 60	$\substack{+\\+}70$	$\substack{-\\-}20$	± 30	∓ 80	$\substack{-\\-}50$	$+0$
135	-0	± 45	$+90$	$\substack{-\\-}45$	$+0$	∓ 45	-90	$\substack{+\\+}45$	-0
140	± 40	∓ 80	$\substack{-\\-}60$	$\substack{+\\+}20$	∓ 20	± 60	$\substack{+\\+}80$	$\substack{-\\-}40$	$+0$
145	∓ 80	$\substack{-\\-}65$	$\substack{+\\+}30$	∓ 5	± 40	∓ 75	$\substack{-\\-}70$	$\substack{+\\+}35$	-0
150	$\substack{-\\-}60$	$\substack{+\\+}30$	-0	± 30	∓ 60	-90	$\substack{+\\+}60$	$\substack{-\\-}30$	$+0$
155	$\substack{+\\+}20$	∓ 5	± 30	∓ 55	± 80	$\substack{+\\+}75$	$\substack{-\\-}50$	$\substack{+\\+}25$	-0
160	∓ 20	± 40	∓ 60	± 80	$\substack{+\\+}80$	$\substack{-\\-}60$	$\substack{+\\+}40$	$\substack{-\\-}20$	$+0$
165	± 60	∓ 75	-90	$\substack{+\\+}75$	$\substack{-\\-}60$	$\substack{+\\+}45$	$\substack{-\\-}30$	$\substack{+\\+}15$	-0
170.	$\substack{+\\+}80$	$\substack{-\\-}70$	$\substack{+\\+}60$	$\substack{-\\-}50$	$\substack{+\\+}40$	$\substack{-\\-}30$	$\substack{+\\+}20$	$\substack{-\\-}10$	$+0$
175	$\substack{-\\-}40$	$\substack{+\\+}35$	$\substack{-\\-}30$	$\substack{+\\+}25$	$\substack{-\\-}20$	$\substack{+\\+}15$	$\substack{-\\-}10$	$\substack{+\\+}5$	-0
180	$+0$	-0	$+0$	-0	$+0$	-0	$+0$	-0	$+0$

CHAPTER IV

TREATMENT OF NON-SINUSOIDAL ALTERNATING QUANTITIES. THE USE OF THE OSCILLOGRAPH

43. General. — Following out the plan, as stated, of considering some of the more important matters relating to special investigation of transmission line phenomena, this chapter, which deals with the methods of obtaining the periodic curve and of utilizing the results of harmonic analysis of the same, is included.

For a large part of alternating current investigation the complex quantity method of treatment, to which such prominence has been given by the work of Steinmetz, is the ideal one. A very brief outline of the method is sufficient here since detailed treatment is to be found in most all of the modern texts.

The simplest alternating quantity may be described as one which varies harmonically with time between numerically equal positive and negative limits. Such a variation is represented algebraically by a trigonometric function — either cosine or sine, or both, depending upon the arbitrarily selected instant from which to count time. Thus, the displacement of a clock pendulum (whose motion is approximately harmonic) from its position of equilibrium would be represented by a cosine function of time if the zero of time was chosen as the moment when the pendulum was in an extreme position away from its equilibrium position; while if the zero of time was taken at the instant the pendulum passes through its equilibrium position, the displacement therefrom is represented by a sine function of time. This applies equally to alternating electrical quantities, in that we may assign to one of the quantities in the system either a cosine or a sine (or a combination of both) variation by the proper selection of our arbitrary zero of time. All other quantities must be referred to this same zero, and thus such an arbitrary selection of the particular trigonometric function for one of the quantities can be made for this one quantity only — all others being represented by such functions as will give to them their actual phase displacement from the

initial quantity. By properly selecting the zero of time important simplifications may often be made in the *form* of the analytic expressions for physical phenomena.

44. The Polar Diagram and Vector Representation of Alternating Quantities. — Let the alternating current flowing in an electrical circuit be represented by

$$i = a \cos \omega t. \tag{1}$$

Plotted in rectangular coördinates there results the familiar cosine curve. Plotted in polar coördinates with OX taken as the

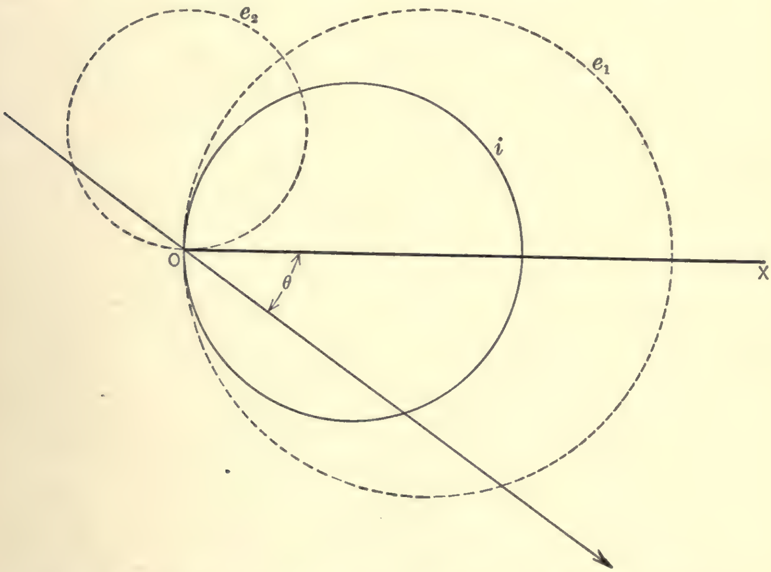

Fig. 12. — The Circles Representing Trigonometric Functions Plotted in Polar Coördinates. OX is the Reference Line.

reference line (Fig. 12) and with the angle $\theta = \omega t$ as independent variable counted positive in the clockwise direction, the curve becomes a circle whose diameter lies along the reference line OX and has a length a. This arbitrary selection of positive direction of rotation is opposite to that usually used in mathematical discussions. It is used thus in order to lead to a standardized form of expression for impedances.

Suppose the current i to flow through a circuit containing

resistance, inductance, and capacity; r, L, and C, respectively. Then by the fundamental laws of the electric circuit, the impressed e.m.f. at any instant is

$$e = ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt, \tag{2}$$

which, for a current flow represented by equation (1), becomes

$$e = ar\cos\omega t - a\left(\omega L - \frac{1}{\omega C}\right)\sin\omega t. \tag{3}$$

The e.m.f. thus consists of two components, a cosine and a sine component when referred to the same zero instant as the current i. The two dotted circles in Fig. 12 represent these two components of e, where

$$\left.\begin{aligned} e &= e_1 + e_2, \\ e_1 &= ar\cos\omega t, \\ e_2 &= -\,a\left(\omega L - \frac{1}{\omega C}\right)\sin\omega t. \end{aligned}\right\} \tag{4}$$

The single curve representing e as the resultant or sum of the two circles, e_1 and e_2, has its diameter displaced from the horizontal by an angle whose tangent is

$$-\frac{\omega L - \dfrac{1}{\omega C}}{r}.$$

Fig. 13 shows the two circles, e and i.

For convenience, the circles may be omitted from the diagram and the diameters only, retained. Further, since, in general, effective or root-mean-square values are desired, the diameters may be drawn to such a scale as to represent these values which are $\frac{1}{\sqrt{2}}$ times the maximum. Such a representation of alternating quantities constitutes the so-called "vector diagrams." They are in no sense vector diagrams, however, since the lines drawn therein are, in effect, only the diameters of the polar circles which have arbitrarily been omitted, and do not possess any of the properties of a physical "vector." The only point of resemblance lies in the fact that the parallelogram law of combination of such diameters holds true for different components of the *same* alternating quan-

tity. Since the term "vector diagram" has become so familiar and receives such wide usage, it appears proper to retain it here, the distinction between such diagrams and the true vector diagram of mechanics being recognized, however.

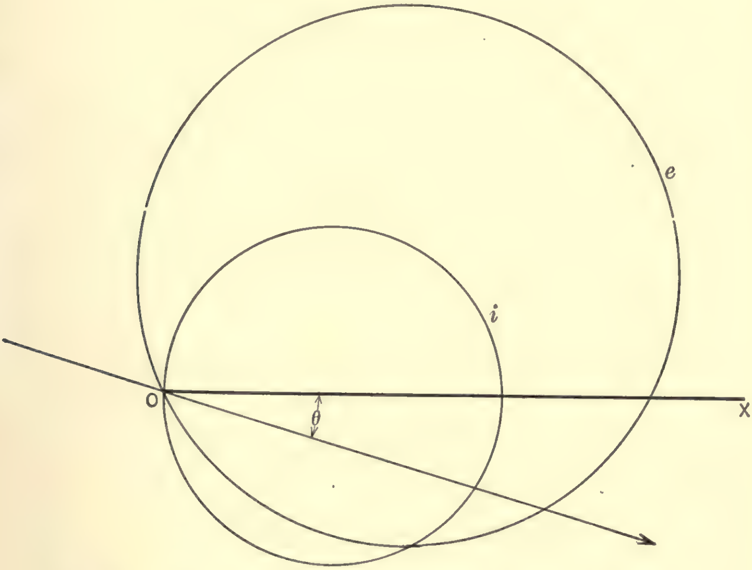

FIG. 13. — The Polar Circles Representing Voltage and Current.

45. Complex Quantity Representation. — Having reduced the representation of trigonometric functions to the simple straight lines of the vector diagram, further simplification is to be made. The positions of the ends of the vectors in the diagram may be specified in a system of rectangular coördinates, and the diameters of the polar circles which are represented by these vectors are completely located thereby. In naming the coördinates of the end of the vector, horizontal distances are taken positive to the right and negative to the left of the origin. Distances in a perpendicular direction are positive when above and negative when below the horizontal axis. To distinguish these latter components from the horizontal, the prefix j is used.

According to this convention, the two alternating quantities, i and e, as shown in Fig. 13 by their polar circles and in Fig. 14 by

the corresponding vectors, and whose equations in time are given by (1) and (3), may be most conveniently expressed by

$$\left.\begin{aligned} I &= a + j0, \\ E &= ar + ja\left(\omega L - \frac{1}{\omega C}\right). \end{aligned}\right\} \qquad (5)$$

In (5), capital letters are used, indicating an alternating quantity specified in vector notation. Cosine functions are always repre-

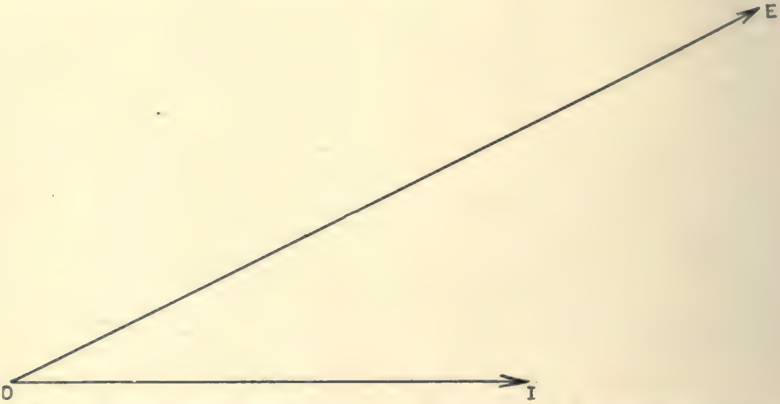

FIG. 14. — The Vectors Representing the Diameters of the Polar Circles.

sented by the term not affected with the symbol j. A sine wave of the form

$$e_2 = -a\left(\omega L - \frac{1}{\omega C}\right)\sin \omega t$$

becomes

$$E_2 = +ja\left(\omega L - \frac{1}{\omega C}\right)$$

in the new notation.

Negative sine functions take the $+j$ as prefix, and positive sine functions the $-j$, while the cosine functions require no prefix and retain their original algebraic sign when expressed in the new notation.

Alternating quantities are thus expressible in either of the two analytical forms — by trigonometric functions, or by means of the simplified notation above. The vector $C = a + jb$ may

immediately be written $c = a \cos \omega t - b \sin \omega t$, unless effective values are represented by the vector notation, in which case, to give true instantaneous values, we must write

$$c = \sqrt{2}\,(a \cos \omega t - b \sin \omega t).$$

46. The Complex Operator — Ratio between Two Alternating Quantities Expressed as Vectors. — The method of expression of alternating quantities by the use of so-called vectors, and the simplified notation for such, involving the use of the prefix j, does not imply that any significance be attached to the symbol j other than that of a mere distinguishing mark to designate a distance above or below the reference axis in the polar diagram. Great use is made, however, of a further assigned significance.

In equation (5) the ratio between e.m.f. E and current I may be formed, E and I retaining their vector expression.

$$Z = \frac{E}{I} = \frac{ar + ja\left(\omega L - \dfrac{1}{\omega C}\right)}{a + j0} \tag{6}$$

serves as a defining equation for the quantity Z, which must have the same physical dimensions as electrical resistance, being the ratio of voltage to current. By division in (6)

$$Z = r + j\left(\omega L - \frac{1}{\omega C}\right). \tag{7}$$

This quantity Z cannot represent any alternating quantity of a type similar to E or I, for if the ratio of instantaneous values is taken

$$\frac{e}{i} = \frac{ar \cos \omega t - a\left(\omega L - \dfrac{1}{\omega C}\right)\sin \omega t}{a \cos \omega t} = r - \left(\omega L - \frac{1}{\omega C}\right)\tan \omega t$$

and this ratio passes recurrently through positive and negative infinite values, while writing (7) in a trigonometric form by the transformation defined in the preceding paragraph yields

$$r \cos \omega t - \left(\omega L - \frac{1}{\omega C}\right)\sin \omega t$$

which is a totally different result, and therefore wrong.

The quantity Z as defined by (7) is merely an *operator*. Multiplication of the current value I by this operator yields the vector

expression for e.m.f., or inversely, division of the voltage expression E, in vector form, by this operator yields the expression for current, I. To be such an operator, this quantity Z must be independent of the arbitrary selection of the zero instant of time. This property may be made the basis for an interpretation of the significance of the symbol j.

Take the origin of time so that the expression for current as given in equation (1) becomes

$$i = a \sin \omega t.$$

Then

$$e = ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt$$
$$= ar \sin \omega t + a\left(\omega L - \frac{1}{\omega C}\right)\cos \omega t,$$

and in the vector notation

$$I = 0 - ja,$$
$$E = a\left(\omega L - \frac{1}{\omega C}\right) - jra,$$

from which the ratio

$$Z = \frac{E}{I} = \frac{a\left(\omega L - \frac{1}{\omega C}\right) - jra}{-ja}. \tag{8}$$

Equating the values of Z as given by (7) and (8),

$$\frac{\left(\omega L - \frac{1}{\omega C}\right) - jr}{-j} = r + j\left(\omega L - \frac{1}{\omega C}\right),$$

from which, after multiplying throughout by $-j$,

$$-j^2 = 1, \qquad \text{or} \qquad j = \sqrt{-1}. \tag{9}$$

The assignment of this numerical significance to the quantity j does not prevent its use as a designating symbol for the upward direction in the vector diagram, while it does enable all formal algebraic operations, multiplication, addition, extraction of roots, etc., incident to computations relative to complicated electrical systems, involving the use of such operators as the complex quantity Z, or its reciprocal Y, to be rigorously carried out. The reader will find full discussion of the use of such complex quantity

representation of impedances, Z, in connection with the vector representation of alternating electrical quantities in any treatise on alternating currents — particularly in Steinmetz' "Alternating Current Phenomena," where the method is developed in great detail and applied to a large number of special cases.

47. Transformation from Trigonometric Functions of Time to Complex Representation. — The points for which emphasis is sought here are: 1st, the ready transformation from a trigonometric expression for an alternating quantity, $a \cos \omega t + b \sin \omega t$, to a vector representation, $a - jb$, and vice versa; and 2nd, the fundamental difference between the vectors used in the diagram which represent alternating quantities, and the ratios between such quantities which are the complex operators and can in no sense be drawn in such a vector diagram, although they are represented algebraically by a similar notation. Thus $E = e_1 + je_2$ represents a definite alternating quantity,

$$e_1 \cos \omega t - e_2 \sin \omega t,$$

and as such may be drawn as a vector in the conventional vector diagram, e_1 and e_2 being the coördinates of the end of the vector, while $Z = r + jx$ represents only the complex operator which signifies the ratio between voltage impressed upon and current flowing in a circuit of resistance r and inductive reactance x.

For discussion of the quantities, resistance, reactance, conductance, and susceptance, as well as the allied quantities, impedance and admittance, the reader is referred to any text on alternating currents.

48. Superposition. Use and Limitations of the Complex Quantity Method. — In any electrical circuit whose physical properties, resistance, self-inductance, capacity, etc., remain constant regardless of the magnitudes of the voltages or currents present, any number of impressed voltages, if they be of the same frequency and pure sinusoidal wave shape, will produce currents and potential differences among the several portions of the system, all of which are of the same frequency and of a pure sinusoidal form. Calculations of phenomena in such circuits are made using the vector notation for alternating quantities and the complex form for the operators which represent the properties of the various portions of the circuit (impedance, admittance, etc.), in exactly the same manner as would be done for a similar combination of

resistances only under the influence of unvarying, unidirectional impressed voltages. Kirchhoff's Laws, with impedance as the generalized term substituted for resistance, form the basis for the development of equations.

If, in such a system in operation, a source of e.m.f. of different frequency be inserted, the calculations relative to the phenomena of the existing frequency still hold true, but in addition to these phenomena there is added a new set of the same frequency as the new e.m.f. whose frequency differs from the other. In other words, the method of superposition is to be employed in making calculations, and, in fact, the phenomena may themselves be regarded as the resultant of several component parts. It must be remembered that the scheme of superposition is merely a conceptual one, however, and that at any instant there is but one actual current or potential difference present in a given portion of a system. It is necessary to realize that not only is the method of superposition, but also the use of vector representation inapplicable, when the properties of a circuit vary with variations in the magnitude of currents or voltages. A case in point is that of an iron cored reactance, where the permeability of the core, and thereby the self-inductance of the circuit when defined in any way whatsoever, is not a constant, but varies with the magnetizing force. In such cases, sinusoidal impressed e.m.f.'s lead to non-sinusoidal currents, and vice versa.

As opposed to the above limitations may be mentioned the case in which the apparent constants or properties of a circuit vary with the changes in frequency, but not with changes in magnitudes. Such cases may be rigorously treated by the complex quantity (vectors and complex operators) method provided care is used in selecting the values of the circuit properties which exist at the frequency under discussion. For example, if a circuit contains an air-cored inductance coil, it would be perfectly proper to base vector quantity calculations for this circuit on the values of effective resistance and inductance of this coil at the frequency in question. In general the effective resistance and effective inductance of a compactly wound coil will change with change in frequency, but these changes arise from the fact that the coil to which we attach the notion of simple resistance and self-inductance really consists of a very intricate combination of inductances, resistances, and capacities (capacities between adjacent turns) and

that the conductor of which it is composed is of finite size and thus has an unequal current distribution over its cross section, so that from the very nature of the case we could not expect the *apparent* resistance and self-inductance to be the same at different frequencies. By regarding such a coil as an infinitely complicated system of simple parts of circuits we would still expect the properties of the coil to be independent of the magnitude of current flow; and such is the case, the effective constants determined for any particular frequency and current strength holding true for all values of current at this same frequency. In such a case the only significance the terms "effective resistance" and "effective self-inductance" can have are respectively the real portion and $\frac{1}{\omega}$ times the reactance, or j term, of the observed complex ratio

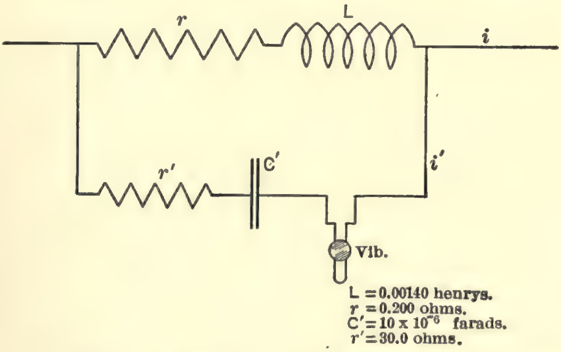

FIG. 15. — Oscillograph Distortion Circuit for Current Measurements.

between impressed voltage and current at the particular frequency used.

49. Numerical Example of Complex Quantity Treatment of Non-Sinusoidal Alternating Quantities. — To illustrate the method of treatment of non-sinusoidal periodic alternating currents by the vector method, application to some special cases will be shown.

Suppose that in order to determine the wave-shape of the current flowing in a given conductor an oscillograph is used, connected as shown in Fig. 15. In this figure i is the current in the main conductor, whose wave shape is to be determined. The oscillograph galvanometer is connected in series with a resistance

r' and a condenser C', the whole being shunted around an inductance coil of resistance r and self-inductance L. The constants of the two circuits are adjusted so that the proper amount of current for the operation of the oscillograph galvanometer is diverted through the same. Reasons for selecting a circuit arrangement as shown will be given later. The upper curve in Fig. 16 shows the wave shape as recorded by the oscillogram. By means of a direct current calibration of the galvanometer, the amount of current per unit linear deflection of the spot of light from the zero-point may be determined, and thus the scale of ordinates may be such as to indicate directly the number of amperes of current flowing in the oscillograph vibrator circuit. The y coördinates of the upper curve in Fig. 16 give instantaneous values of the current i'.

Assuming for the present that the oscillograph records accurately the instantaneous value of current flowing therein, correction must be made for the distorting effect of the circuit arrangement in order to determine the actual wave shape of the current in the main circuit, since it is only when non-inductive resistances are used, both for the shunt and for the vibrator circuit proper, that the wave shape as shown by the oscillogram will be the same as that of the line current.

Analyzing the oscillogram for i' by means of Analysis Forms F and G, from 18 ordinates measured over one-half wave, and with the origin taken as indicated in the figure, the equation of the vibrator current was determined to be

$$\begin{aligned} i' = & -0.0204 \cos\theta + 0.0347 \sin\theta \\ & +0.0210 \cos 3\theta + 0.0018 \sin 3\theta \\ & -0.0034 \cos 5\theta - 0.0072 \sin 5\theta \\ & -0.0038 \cos 7\theta + 0.0037 \sin 7\theta \\ & -0.0052 \cos 9\theta - 0.0002 \sin 9\theta \\ & -0.0003 \cos 11\theta + 0.0058 \sin 11\theta \\ & +0.0001 \cos 13\theta - 0.0003 \sin 13\theta \\ & +0.0004 \cos 15\theta - 0.0003 \sin 15\theta \\ & -0.0004 \cos 17\theta - 0.0005 \sin 17\theta. \end{aligned} \qquad (10)$$

The fundamental frequency is 60 cycles per second. Then, $\omega_n = 2\pi n\,(60) = 377\,n =$ angular velocity for nth harmonic. If

$$Z_n = r + j\omega_n L = \text{impedance of shunt,}$$

and

$$Z_n' = r' - j\frac{1}{\omega_n C'} = \text{impedance of vibrator circuit,}$$

by the application of Kirchhoff's Law it is easily shown that in vector notation

$$I_n = \left(1 + \frac{Z_n'}{Z_n}\right) I_n'. \tag{11}$$

The vector expressions for vibrator current may be written immediately from equation (10); thus, for example, using maximum values,

$$I_3' = 0.0210 - j\,0.0018,$$

while the term in parentheses in (11) is to be computed from the values of the circuit constants. The following table indicates the main features and results of the calculation by equation (11).

TABLE IX

Correction for Oscillograph Distortion Circuit
(Maximum values of current)

n	Z_n'	Z_n	$\left(1+\frac{Z_n'}{Z_n}\right)$	I_n'	I_n
1	$30 - j265$	$0.20 + j0.528$	$-419 \; -j215$	$-0.0204 - j0.0347$	$1.10 + j18.9$
3	$30 - j88.3$	$0.20 + j1.58$	$-51.8 - j25.6$	$0.0210 - j0.0018$	$-1.13 - j0.44$
5	$30 - j53.0$	$0.20 + j2.64$	$-18.1 - j12.8$	$-0.0034 + j0.0072$	$0.154 - j0.087$
7	$30 - j37.9$	$0.20 + j3.70$	$-8.76 - j8.64$	$-0.0038 - j0.0037$	$0.001 + j0.065$
9	$30 - j29.4$	$0.20 + j4.75$	$-4.93 - j6.55$	$-0.0052 + j0.0002$	$0.027 + j0.033$
11	$30 - j24.1$	$0.20 + j5.81$	$-2.97 - j5.30$	$-0.0003 - j0.0058$	$-0.030 + j0.019$
13	$30 - j20.4$	$0.20 + j6.86$	$-1.84 - j4.46$	$0.0001 + j0.0003$	$0.001 - j0.001$
15	$30 - j17.7$	$0.20 + j7.92$	$-1.14 - j3.84$	$0.0004 + j0.0003$	$0.001 - j0.002$
17	$30 - j15.6$	$0.20 + j8.98$	$-0.66 - j3.37$	$-0.0004 + j0.0005$	$0.002 + j0.001$

The complex values of I_n give the following equation which represents the line current i as a function of time when referred to the same zero as the oscillograph current i'.

$$\begin{aligned} i = \; & 1.10 \cos\theta - 18.9 \sin\theta \\ & -1.13 \cos 3\theta + 0.44 \sin 3\theta \\ & +0.154 \cos 5\theta + 0.087 \sin 5\theta \\ & +0.001 \cos 7\theta - 0.065 \sin 7\theta \\ & +0.027 \cos 9\theta - 0.033 \sin 9\theta \\ & -0.030 \cos 11\theta - 0.019 \sin 11\theta \\ & +0.001 \cos 13\theta + 0.001 \sin 13\theta \\ & +0.001 \cos 15\theta + 0.002 \sin 15\theta \\ & +0.002 \cos 17\theta - 0.001 \sin 17\theta. \end{aligned} \tag{12}$$

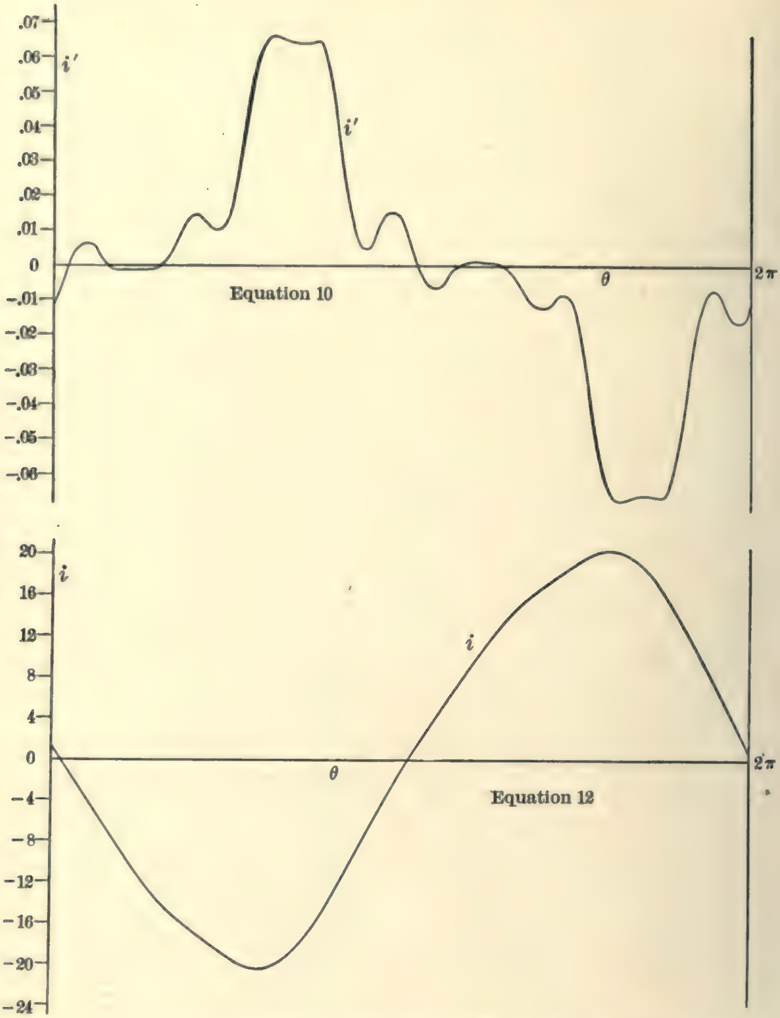

FIG. 16. — The Distortion Produced by the Circuit of Fig. 15. Upper: Oscillograph Vibrator Current, i'. Lower: Main Line Current, i, by Analysis.

The lower curve in Fig. 16 shows the wave represented by equation (12), in the proper phase position with reference to the upper curve which represents the oscillograph vibrator current.

A comparison of the two curves, i and i', should immediately show the advantage of using a circuit connection as illustrated by Fig. 15. The higher harmonics are very much accentuated in the vibrator circuit, and thus in the resulting oscillogram, thereby rendering their determination much more accurate than would be possible if an undistorted wave shape were used. The amount of the distortion is strikingly illustrated by a comparison of the different frequency values of the quantity $1 + \frac{Z'}{Z}$, as given in Table IX, and which represent the ratios of line current to oscillograph current.

50. Distorting Circuits for Use with the Oscillograph. — The oscillograph is used to obtain one of two things — either the wave shape of the current in some conductor, or the wave shape of a potential difference or an electromotive force. These, at any rate, will be the quantities of particular interest in a consideration of transmission phenomena. With the exception of an electrostatic form of oscillograph, all types of oscillographs require for their operation a current of appreciable magnitude, since they are based upon either an electromagnetic or hot-wire principle, and thus they cannot indicate potential differences directly but only through the medium of a current produced thereby. One of the most popular commercial oscillographs requires a current of approximately 50 milliamperes to operate it satisfactorily, so that in many instances the effect of a current flow of this magnitude on the phenomena in the system under investigation cannot be neglected. For the recording of potential differences, the electrostatic form of instrument is theoretically the ideal one — its electrostatic capacity being so small that the current consumed thereby is negligible, except at frequencies far higher than the vibrating system could record. The electromagnetic type, on the other hand, affords almost an ideal instrument for the indication of current, since the resistance of the galvanometer circuit may be made as low as one or two ohms, and which, for the indication of currents of large magnitude, could be connected around a shunt of very low resistance. The whole combination would thus cause a drop in potential of only one-twentieth to one-tenth of a volt, a quantity usually negligible.

In a large number of instances where it is definitely known that the effect of the oscillograph circuit in altering the ex-

isting phenomena in the system can be neglected, or when it is desired to record the phenomena which occur as affected by the oscillograph regardless of whether or not they are the same as would occur with the oscillograph removed, a very great increase in the accuracy of determination of the various frequency components can be secured by so arranging the oscillograph circuit that all of the harmonics whose values are desired produce currents through the galvanometer of the same order of magnitude. In the numerical example just considered this condition was secured by means of the circuit combination shown in Fig. 15. With such a circuit arrangement any particular harmonic may be accentuated in the oscillogram, for by adjusting the values of L and C' a condition approximating current resonance for this particular frequency may be secured, in which case the vibrator current may be made much greater, even, than the total line current. In order to avoid difficulties from current resonance where such is not desired, it is always well to insert considerable resistance in series with the galvanometer and condenser, and while observing the wave shape on the visual screen to reduce this resistance until the desired distortion is obtained. If one frequency predominates to an undesirable extent, a change in the condenser capacity will usually remedy the difficulty. In using any oscillograph circuit containing capacity, it is well to always make the first connection to the circuit under test through a comparatively high resistance which subsequently is gradually removed. This prevents a transient flow of current, into the condenser, of a magnitude sufficient to burn out the protective fuses placed in series with the galvanometer.

In this discussion of distorting circuits to be used for accentuating the higher harmonics, it is of course to be understood that the scheme is to be applied only when the system is operating in a steady condition. It is obvious that an investigation of transient phenomena would not permit of the use of circuits of the type discussed here.

If the extreme distortion possible with the connection shown in Fig. 15 is not desired, a very convenient connection consists in shunting an inductance coil with a non-inductive vibrator circuit — that is, using the connection in Fig. 15 with the condenser C' short-circuited. Another connection which would yield approximately the same result would be that in which the galvanometer

circuit containing a condenser and resistance was shunted around a non-inductive resistance.

In making an oscillogram of the current flowing in a high voltage system, where it would be impracticable to utilize any direct shunt connection on account of the large potential differences, most satisfactory results may be obtained by using a mutual inductance coupling between the high voltage circuit and the oscillograph circuit.* For this purpose, air-core inductance coils of large diameters and small winding sections will permit of a fairly close electromagnetic coupling with a separation between coils

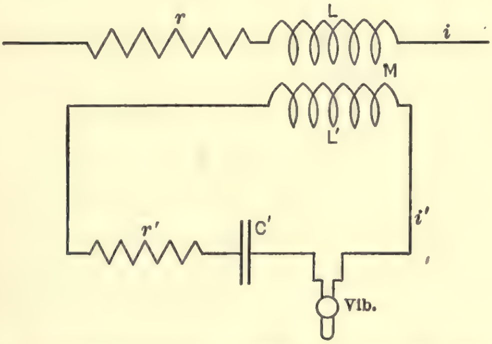

FIG. 17. — Electromagnetically Coupled Current-distorting Circuit.

great enough to eliminate any danger from the high potential system. One of the coils is connected in series with the high voltage circuit and the other is connected in series with the oscillograph vibrator and the proper regulating resistance. If great distortion is desired, the oscillograph circuit may contain a condenser. For a circuit connection as shown in Fig. 17 the following equation expresses the relation between oscillograph and line currents, from which the results may be corrected so as to yield the true wave shape of line current.

$$I_n = \frac{-\left(\omega_n L' - \dfrac{1}{\omega_n C'}\right) + jr'}{\omega_n M} I_n'. \tag{13}$$

* The use of current transformers for this purpose is not recommended if a distortion circuit is to be employed, for the impedance of such a circuit may require a voltage sufficient to prevent a proper current transformation.

Equation (13) is to be used in exactly the same manner as equation (11), illustration of which has been given. If the condenser is not present, that is, equivalent to being short-circuited, the term $1/\omega_n C'$ is zero, in (13).

If the wave shape only of the line current is desired, and not the absolute magnitude, it is obvious from (13) that the value of the mutual inductance, M, need not be known in order to make the reduction. This is a distinct advantage, and in fact for almost any investigation of current wave shape, the form of circuit shown in Fig. 17 is superior to that shown in Fig. 15, since a flexibility of control by variation of the mutual inductance is possible; and further, the labor of computation necessary for reducing the oscillographic data is considerably less by equation (13) than by equation (11). Of course care must be exercised to prevent extraneous varying magnetic fields from affecting the oscillograph circuit, for in such an event, the determination of the line current would be vitiated.

51. Oscillographic Determination of Wave Shape of E.M.F.—To secure oscillograms representing wave shapes of voltage by means of either the electromagnetic or hot-wire type of instrument, the current flow must be limited by some form of series impedance. If non-inductive resistance is used, the wave shape of the current through the vibrator will be identical with that of the impressed voltage at the time of recording, though on account of the current consumed this voltage wave shape may differ from that which would exist on removal of the instrument. If distortion of wave shape is desired, the current flow through the galvanometer may be limited by inserting series capacity, and at the same time decreasing the resistance. If the resistance present be small in comparison with the impedance of the condenser, the distortion secured by this scheme of connection will be approximately proportional to the order of the harmonic. This is apparent from the fact that the impedance of a condenser changes in inverse proportion to change in frequency.

Very much greater distortion may be secured by the use of a circuit of the type shown in Fig. 18. In this diagram, C and r constitute the chief current limiting impedance, while the parallel circuits, C', r' and L'', r'', are for the purpose of securing a selective current flow through the galvanometer, and may be of much lower impedance than the circuit consisting of C and r. The equation

necessary for the determination of the line voltage E from the current I' as represented by the oscillogram is

$$E_n = \left(Z_n + Z_n' + \frac{Z_n Z_n'}{Z_n''}\right) I_n' \tag{14}$$

where Z_n, Z_n', and Z_n'' are the impedances for the nth harmonic of the respective portions of the circuit.

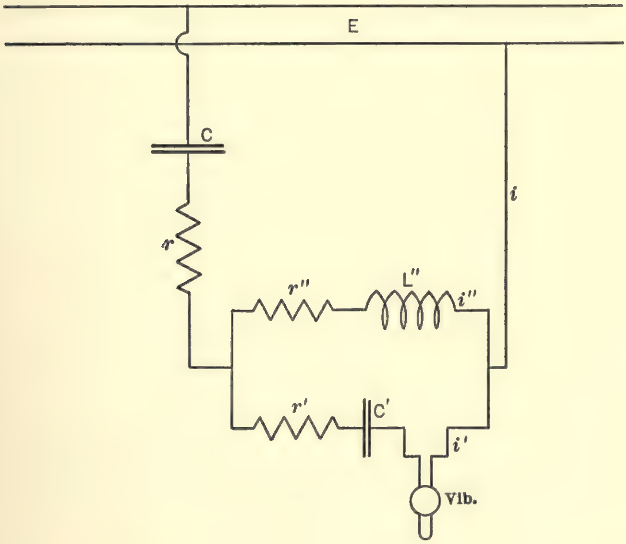

FIG. 18. — Distortion Circuit for Determining Voltage Wave Shapes.

The current consumed by the entire circuit arrangement is

$$I_n = \left(1 + \frac{Z_n'}{Z_n''}\right) I_n'. \tag{15}$$

To illustrate the use of such a circuit the following numerical example is considered. In this case the vibrator current wave shape is an assumed one, the values of the various frequency components having been selected at random, but with care that the total vibrator current be of the proper magnitude. Even harmonics are included.

The equation for the vibrator current i' is

$$\begin{aligned} i' = \; & 0.0158 \cos \theta - 0.0054 \sin \theta \\ & -0.0060 \cos 2\theta + 0.0058 \sin 2\theta \\ & -0.0101 \cos 3\theta + 0.0054 \sin 3\theta \\ & +0.0014 \cos 4\theta + 0.0015 \sin 4\theta \\ & -0.0026 \cos 5\theta - 0.0112 \sin 5\theta \\ & +0.0004 \cos 6\theta - 0.0064 \sin 6\theta \\ & +0.0079 \cos 7\theta + 0.0022 \sin 7\theta, \end{aligned} \tag{16}$$

and the wave shape represented thereby is shown by the upper curve of Fig. 19.

Let the circuit constants shown in Fig. 18 have the following numerical values.

$r = 200.0$ ohms. $\qquad C = 0.50 \times 10^{-6}$ farads.
$r' = 12.0$ ohms. $\qquad C' = 9.0 \times 10^{-6}$ farads.
$r'' = 10.0$ ohms. $\qquad L'' = 0.150$ henrys.

Table X shows the principal numerical values entering into the transformation from the observed harmonics in the oscillogram, that is in i', to those of voltage e.

TABLE X

DETERMINATION OF VOLTAGE WAVE SHAPE FROM OSCILLOGRAPH VIBRATOR CURRENT. CONNECTIONS AS PER FIG. 18
(Frequency of fundamental, 60 cycles)

n	1	2	3	4	5	6	7
Z_n	200 $-j\,5305$	200 $-j\,2652$	200 $-j\,1768$	200 $-j\,1326$	200 $-j\,1061$	200 $-j\,884$	200 $-j\,758$
Z_n'	12.0 $-j\,294.7$	12.0 $-j\,147.4$	12.0 $-j\,98.2$	12.0 $-j\,73.7$	12.0 $-j\,58.9$	12.0 $-j\,49.1$	12.0 $-j\,42.1$
Z_n''	10.0 $+j\,56.5$	10.0 $+j\,113$	10.0 $+j\,169$	10.0 $+j\,226$	10.0 $+j\,282$	10.0 $+j\,339$	10.0 $+j\,396$
$\frac{E_n}{I_n}$	-6620 $+j\,20{,}900$	-630 $+j\,574$	-87 $-j\,876$	$+58$ $-j\,986$	$+118$ $-j\,912$	$+148$ $-j\,814$	$+166$ $-j\,727$
I_n	0.0158 $+j0.0054$	-0.0060 $-j0.0058$	-0.0101 $-j0.0054$	0.0014 $-j0.0015$	-0.0026 $+j0.0112$	0.0004 $+j0.0064$	0.0079 $-j0.0022$
E_n	-218 $+j\,294$	7.11 $+j0.22$	-3.86 $+j9.32$	-1.40 $-j1.47$	9.91 $+j3.69$	5.28 $+j0.62$	-0.29 $-j6.11$

(Slide rule accuracy in computations.)

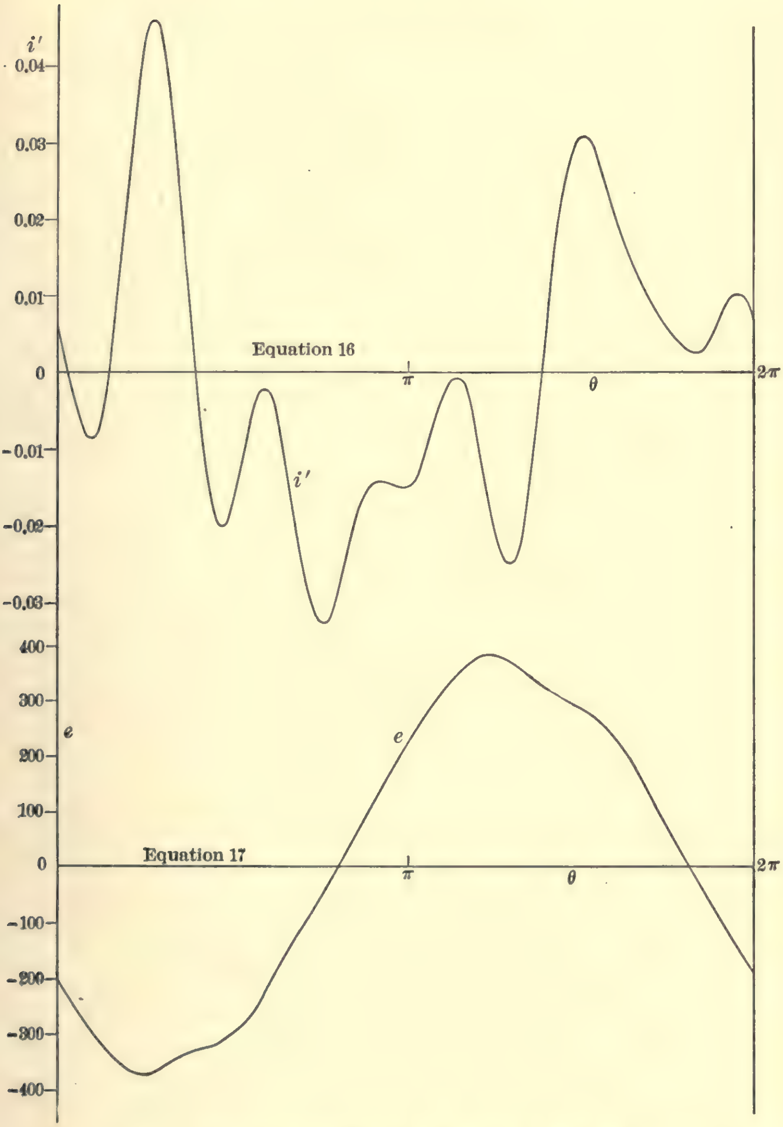

Fig. 19. — The Distortion Produced by the Circuit of Fig. 18. Upper: Oscillograph Vibrator Current, i'. Lower: Main Line Voltage, e, by Analysis.

The equation for instantaneous values of e is then, from Table X,

$$\begin{aligned} e = & -218 \cos\theta - 294 \sin\theta \\ & +7.11 \cos 2\theta - 0.22 \sin 2\theta \\ & -3.86 \cos 3\theta - 9.32 \sin 3\theta \\ & -1.40 \cos 4\theta + 1.47 \sin 4\theta \\ & +9.91 \cos 5\theta - 3.69 \sin 5\theta \\ & +5.28 \cos 6\theta - 0.62 \sin 6\theta \\ & -0.29 \cos 7\theta + 6.11 \sin 7\theta. \end{aligned} \tag{17}$$

The lower curve in Fig. 19 was plotted point by point from equation (17), and serves very well to illustrate the magnifying effect of the circuit used on the higher harmonics. It should be noticed that for the second and third harmonics, the distortion is particularly great on account of the approximation of these frequencies to the resonant frequency of the circuits Z' and Z''. This is indicated by the low values of the ratio $\frac{E}{I}$ in Table X. All of the different frequency components of higher order than the first are magnified to about thirty times their normal values in the oscillogram. Using a smaller condenser capacity C', or a smaller self-inductance L'', would prevent the great relative magnification of the second and third harmonics, but would still magnify those of higher order to a sufficient extent.

CHAPTER V

EQUATIONS OF MOTION OF THE GALVANOMETER MIRROR IN THE ELECTROMAGNETIC TYPE OF OSCILLOGRAPH

52. General. — In this chapter the equations of motion of a mirror, attached to the two supporting strips which constitute the galvanometer coil of the electromagnetic type of oscillograph are developed, with a view to showing the extent to which the indications of such a device are in error when used to record the wave forms of uniformly alternating currents. The discussion also affords a good illustration of the use of the complex quantity notation as applied to alternating quantities other than currents and electromotive forces.

In using the oscillograph to record continuously alternating currents, there is probability of two kinds of error, viz., change in calibration constant of the vibrator for different frequencies, and a time-phase displacement of the mirror deflection from the current flowing in the supporting strips.

Departure of the damping force acting upon the mirror and strip from strict proportionality with the velocity of these parts would cause the mirror motion to be other than a pure sine wave even though the deflecting force be such. Because of the low velocity of these two parts, however, it is improbable that such an effect could be detected. In calculations relative to the mirror motion, the damping force will be assumed proportional to the velocity.

If the moving system is not damped at all, the deflections of the mirror will at all times be in phase with the deflecting force for frequencies below the natural or free frequency of vibration, while for frequencies above this value, the displacement will reverse in phase. On account of the damping, the deflections will not be in exact phase coincidence with the force acting, but will, for frequencies below the free vibration frequency, be lagging with respect thereto.

The inertia of the moving parts tends to cause a change in the calibration constant of the vibrator — particularly for frequencies approaching the natural frequency.

At very high frequencies it is conceivable that the motion of the vibrator strips in the intense magnetic field produces an induced electromotive force of a magnitude comparable with that impressed upon the vibrator circuit. This matter will be referred to again.

53. Fundamental Equations of Motion of the Vibrator Strips. — Assume the entire length of the vibrator strip to be acted upon by a force

$$a \cos \omega t$$

per unit of length, by virtue of a current

$$i \cos \omega t$$

flowing in the strip. Let

T = tension in the strip, in dynes, and
m = mass of the strip per unit length, in grams.

The fundamental differential equation of motion when the strip is immersed in a viscous liquid is

$$T \frac{\partial^2 s}{\partial x^2} - k \frac{\partial s}{\partial t} + a \cos \omega t = m \frac{\partial^2 s}{\partial t^2}, \tag{1}$$

where x is measured along the equilibrium position of the strip, and s is measured perpendicularly thereto, and thus represents the displacement. k represents the damping force per unit length of strip when the strip is moving with unit velocity.

The solution of the above partial differential equation will yield a combination of decreasing exponential functions (decreasing with time) and trigonometric functions, so that under steady conditions after the transient terms have become zero the motion of the strip at any point in its length will be represented by the trigonometric terms which are not affected by the decreasing exponentials. Thus

$$s = A \cos \omega t + B \sin \omega t, \tag{2}$$

where the integration constants A and B are functions of x and the physical properties of the system.

Since s is a simple harmonically varying function it may be

represented by the conventional complex quantity notation, and the variable t thus eliminated.

Thus, for any arbitrarily selected origin of time,

$$\left.\begin{aligned}F &= \text{deflecting force} = f_1 + jf_2,\\ S &= \text{displacement of strip at any point} = s_1 + js_2,\\ V &= \frac{dS}{dt} = j\omega S = \text{velocity},\\ A &= \frac{dV}{dt} = \frac{d^2S}{dt^2} = -\,\omega^2 S = \text{acceleration}.\end{aligned}\right\} \tag{3}$$

Equation (1) becomes for the permanent condition of operation

$$\left.\begin{aligned}T\frac{d^2S}{dx^2} - kV + F &= mA,\\ T\frac{d^2S}{dx^2} - jk\omega S + F &= -m\omega^2 S.\end{aligned}\right\} \tag{4}$$

Combining terms and placing

$$\left.\begin{aligned}\alpha^2 &= \frac{m\omega^2 - jk\omega}{T},\\ N &= \frac{F}{T},\end{aligned}\right\} \tag{5}$$

equation (4) reduces to

$$\frac{d^2S}{dx^2} + \alpha^2 S = -N, \tag{6}$$

the solution for which is

$$S = A_1\epsilon^{j\alpha x} + A_2\epsilon^{-j\alpha x} - \frac{N}{\alpha^2}, \tag{7}$$

in which A_1 and A_2 are arbitrary integration constants which must be determined from known boundary or initial conditions. Since α^2 is itself complex, the quantity $j\alpha$ is complex, and therefore the exponentials become combinations of trigonometric functions and exponentials with real exponents.

54. Determination of Integration Constants. Their Dependence upon the Moment of Inertia of the Mirror. — Let

$$\left.\begin{aligned}2\,l &= \text{total length of vibrator strip, or}\\ l &= \text{distance from support at end to mirror at}\\ &\quad\ \text{the mid-point of the strips.}\end{aligned}\right\} \tag{8}$$

At $x = l$, the slope of the strip at any instant must be such as to give to the point of contact with the mirror the same acceleration as (7) would yield for $x = l$; this on the assumption that the mirror causes no damping over and above that due to the vibrator strips, and further that the length of the mirror along the strip is negligible as well as any forces due to flexure of the strip at the point of contact. These conditions are not realized exactly, but to take them into consideration would hopelessly complicate the problem. It is not to be expected that the neglect of these matters will vitiate the solution to any great extent.

Let

$$\left.\begin{aligned} 4\,I &= \text{polar moment of inertia of the mirror about a gravity axis parallel to the strips.} \\ 2\,d &= \text{spacing of the two supporting strips, center to center.} \end{aligned}\right\} \tag{9}$$

The normal acceleration at the point of contact is d times the angular acceleration of the mirror, and the torque exerted by each half-strip is

$$-T\,d\left.\frac{dS}{dx}\right]_{x=l}.$$

The algebraic sign used in this expression for torque applies to the first half-strip only, but as all of the four half-strips exert equal accelerating forces upon the mirror, each may be thought of as accelerating one-fourth of the mirror, and our investigation confined thereby to one half-strip only.

$$\left.\frac{d^2s}{dt^2}\right]_{x=l} = -\frac{T\,d^2}{I}\left.\frac{ds}{dx}\right]_{x=l}. \tag{10}$$

In the complex quantity notation, $\frac{d^2s}{dt^2} = -\omega^2 S.$

Let

$$\left.\begin{aligned} H &= \frac{T\,d^2}{I}\text{, and then at } x = l, \\ -\omega^2 S &= -H\frac{dS}{dx};\quad \frac{dS}{dx} = S\frac{\omega^2}{H}. \end{aligned}\right\} \tag{11}$$

Equation (11) serves as one boundary condition for the determination of the integration constants, in that it expresses the

value of the first derivative of the function in terms of the function itself, at a given value of x.

As a second condition, since there are two constants to be determined,

$$S = 0 \text{ at } x = 0.$$

Thus

$$A_1 + A_2 - \frac{N}{\alpha^2} = 0. \tag{12}$$

Substituting (7) in (11), and then placing $x = l$,

$$\frac{\omega^2}{H} A_1 \epsilon^{j\alpha l} + \frac{\omega^2}{H} A_2 \epsilon^{-j\alpha l} - \frac{\omega^2 N}{\alpha^2 H} = j\alpha A_1 \epsilon^{j\alpha l} - j\alpha A_2 \epsilon^{-j\alpha l},$$

or

and

$$\left.\begin{aligned} \left(\frac{\omega^2}{H} - j\alpha\right)\epsilon^{j\alpha l} A_1 + \left(\frac{\omega^2}{H} + j\alpha\right)\epsilon^{-j\alpha l} A_2 &= \frac{\omega^2 N}{\alpha^2 H}, \\ A_1 \qquad\qquad + \qquad\qquad A_2 &= \frac{N}{\alpha^2}. \end{aligned}\right\} \tag{13}$$

Solving the two equations of (13) for A_1 and A_2,

$$\left.\begin{aligned} A_1 &= \frac{N}{\alpha^2} \frac{\dfrac{\omega^2}{H} - \left(\dfrac{\omega^2}{H} + j\alpha\right)\epsilon^{-j\alpha l}}{\left(\dfrac{\omega^2}{H} - j\alpha\right)\epsilon^{j\alpha l} - \left(\dfrac{\omega^2}{H} + j\alpha\right)\epsilon^{-j\alpha l}}, \\ A_2 &= \frac{N}{\alpha^2} \frac{\left(\dfrac{\omega^2}{H} - j\alpha\right)\epsilon^{j\alpha l} - \dfrac{\omega^2}{H}}{\left(\dfrac{\omega^2}{H} - j\alpha\right)\epsilon^{j\alpha l} - \left(\dfrac{\omega^2}{H} + j\alpha\right)\epsilon^{-j\alpha l}}. \end{aligned}\right\} \tag{14}$$

55. The Mirror Motion. — If these values of A_1 and A_2 be substituted in (7), the equation of motion for the strip at any point will be obtained. Since it is the motion of the mirror only that is of interest at present, the introduction of (14) in (7) and the substitution of l for x in the resulting equation will give the desired solution. S, as given by (7), used with the subscript m designates the mirror motion so obtained.

$$S_m = \frac{N}{\alpha^2}\left\{\frac{\dfrac{\omega^2}{H}\left(\epsilon^{j\alpha l} - \epsilon^{-j\alpha l}\right) - 2\,j\alpha}{\dfrac{\omega^2}{H}\left(\epsilon^{j\alpha l} - \epsilon^{-j\alpha l}\right) - j\alpha\left(\epsilon^{j\alpha l} + \epsilon^{-j\alpha l}\right)} - 1.\right\} \tag{15}$$

Writing (15) in a trigonometric form, that is, substituting sines and cosines for the imaginary exponentials,

$$S_m = \frac{N}{\alpha^2}\left[\frac{\dfrac{\omega^2}{H}\sin\alpha l - \alpha}{\dfrac{\omega^2}{H}\sin\alpha l - \alpha\cos\alpha l} - 1\right]$$

which after simplification becomes

$$S_m = \frac{N}{\alpha}\left[\frac{\cos\alpha l - 1}{\dfrac{\omega^2}{H}\sin\alpha l - \alpha\cos\alpha l}\right], \tag{16}$$

which may be taken as the final form.

Since α^2 is complex, α is also complex, and may be written

$$\alpha = u + jv.$$

The square root is most easily formed by expressing α^2 as a scalar with its angle, and then taking for the scalar value of α the square root of the scalar value of α^2, and for the angle of α, one-half the angle of α^2. Thus, if

$$\alpha^2 = a + jb = c\underline{/2\beta},$$

where

$$c^2 = a^2 + b^2,$$

and

$$\tan 2\beta = \frac{b}{a},$$

we have

$$\alpha = \sqrt{c}\underline{/\beta} = \sqrt{c}\cos\beta + j\sqrt{c}\sin\beta.$$

The trigonometric functions of the complex, α, are

$$\left.\begin{aligned}\sin(u + jv) &= \cosh v \sin u + j \sinh v \cos u,\\ \cos(u + jv) &= \cosh v \cos u - j \sinh v \sin u.\end{aligned}\right\} \tag{17}$$

Further reduction of (16) by introducing (17) is not desirable, except numerically, since the resulting expressions become too involved.

56. Application of Analysis to a Particular Case. — To illustrate the application of the foregoing development and to show the magnitude of the errors which may be expected in practice, we will consider the motion of the mirror in a vibrator used by the General Electric Company in their commercial form

of oscillograph. The dimensions of the moving parts are, approximately:

Silver alloy strips, 0.0070″ × 0.00075″.
Total length of strips, 0.438″.
Tension per strip, 3.0 oz.
Spacing of strips, 0.012″.
Glass mirror (thickness increased by 25 per cent to allow for cement used in attaching to strips),

0.060″ × 0.017″ × 0.0075″ thick.

Using 10.0 as the specific weight for the silver strips, and 3.0 for the glass mirror, the following values in C.G.S. units are obtained for the various constants.

$l = 0.555$ cm.
$m = 0.340 \times 10^{-3}$ grams per cm. length.
$T = 83{,}500$ dynes.
$I = 1.75 \times 10^{-8}$ gram-cm².
$d = 0.01525$ cm.

The quantity k is not known, and its calculation from the known dimensions of the system and the viscosity of the liquid would be very difficult, but if the assumption is made that the vibrator strips are critically damped for the fundamental frequency of free vibration, its value may be determined.

In Byerly's "Fourier's Series and Spherical Harmonics," the motion of a string of finite length vibrating in a resisting medium is discussed, and in his solution (on pages 113–115) we find, using the notation adopted here, the angular velocity of free vibration of fundamental frequency to be

$$\omega_1 = \frac{1}{2}\sqrt{\frac{T\pi^2}{ml^2} - \frac{k^2}{m^2}}, \tag{18}$$

so that the free vibration frequency when undamped would be

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{4\,l}\sqrt{\frac{T}{m}}, \tag{19}$$

the familiar equation for vibration of such a type.

For critical damping, by (18),

$$k_1 = \frac{\pi}{l}\sqrt{mT}. \tag{20}$$

For the numerical case in hand, $k_1 = 30.16$ and $f_0 = 7050$. These values are, of course, not exact, since the mass of the mirror has been neglected, but they serve as a basis for an estimate as to the value of k. The free frequency of vibration as given by the manufacturers' data is 5000.

Using the above tabulated numerical constants, three separate calculations for various frequencies have been made, taking three different values for k, viz.,

$$k = 0,$$
$$k = 0.75\,k_1 = 22.62,$$
$$k = 1.50\,k_1 = 45.24,$$

the results of which are given in the following table. The quantity $\frac{S_m}{N}$, as given in the table is proportional to the deflection of the mirror when unit current is flowing in the strips, and thus represents the calibration constant of the galvanometer. The angle ϕ is the lag in time-phase position of the mirror deflection behind that of the current.

TABLE XI

MOTION OF OSCILLOGRAPH VIBRATOR

f	$k = 0$		$k = 0.75\,k_1$		$k = 1.50\,k_1$	
	$\frac{S_m}{N}$	ϕ deg.	$\frac{S_m}{N}$	ϕ deg.	$\frac{S_m}{N}$	ϕ deg.
0	0.1540	0	0.1540	0	0.1540	0
100	0.1541	0	0.1540	1.24	0.1539	2.48
500	0.1556	0	0.1547	6.29	0.1521	12.45
1000	0.1605	0	0.1567	12.78	0.1463	24.53
1500	0.1693	0	0.1599	19.66	0.1389	35.80
2000	0.1834	0	0.1641	27.16	0.1297	46.27
2500	0.2050	0	0.1687	35.45	0.1201	55.82

Figure 20 shows the curves plotted from the above data.

The results of this numerical analysis are very interesting, and substantiate the statements made in the opening of this discussion, section 52. In no case can the oscillograph depict the true wave shape of a complex wave of current in the vibrator strips. If the damping is zero, the deflections are in their proper time-

phase position, but the calibration constant differs for the different frequencies — the higher frequency components being magnified in the oscillogram. For a damping a little greater than 0.75 k_1, the calibration constant of the vibrator, $\frac{S_m}{N}$, will be practically unvarying with change of frequency, but in any case with damping

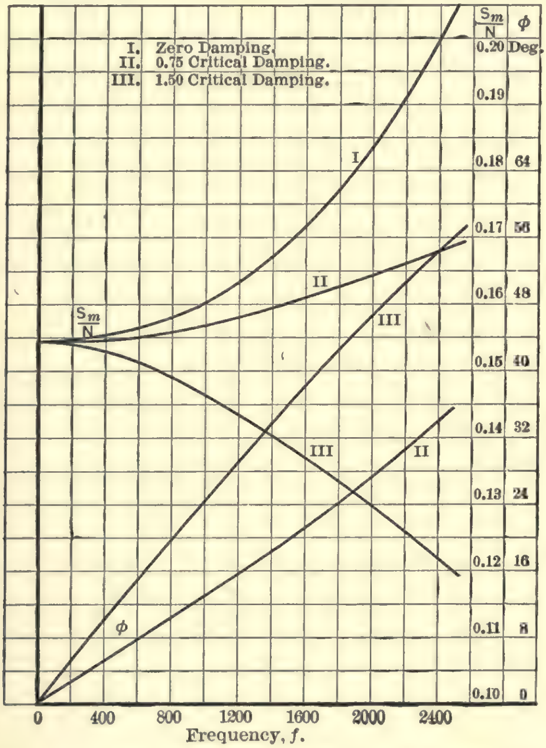

FIG. 20. — Characteristics of Oscillograph Vibrator Motion. Calibration Constant and Angle of Lag from Equation (16).

present, an error in time-phase position occurs. The use of distorting circuits for the purpose of accentuating the higher harmonics in no way increases the percentage error introduced by the oscillograph, for such error is dependent upon the frequency only, and not upon the magnitude of the deflection (of course within the limits of permissible vibrator current).

57. The Intensity of the Magnetic Field Surrounding the Strips. — From the observed calibration constant of the instrument and the computed value of $\frac{S_m}{N}$ it is possible to determine the flux density of the magnetic field in which the vibrator is placed.

Let

L = distance from mirror to observing screen.

c = observed deflection of light spot per absolute unit of current in the galvanometer strips.

I = current in vibrator strips, abamperes.

Then

$$D = cI = \text{deflection of light spot on screen.} \tag{21}$$

For the small deflections used,

$$\frac{D}{2L} = \frac{cI}{2L} = \frac{S_m}{d} = \text{angular rotation of mirror.} \tag{22}$$

From equation (22),

$$\frac{S_m}{N} = \frac{dcI}{2LN}, \quad \text{or} \quad N = \frac{dcI}{2L\frac{S_m}{N}}. \tag{23}$$

By equation (5), however, $N = \frac{F}{T}$, so that in (23)

$$F = \text{force per unit length} = \frac{dcIT}{2L\frac{S_m}{N}} = BI, \tag{24}$$

where B = flux density surrounding the vibrator strips.

Thus

$$B = \frac{dcT}{2L}\left(\frac{N}{S_m}\right). \tag{25}$$

For the vibrator in question, the calibration constant was given as

0.0060 amperes per mm. deflection,

from which

c = 167 cm. per abampere of current.

Also,

L = 46 cm., approximately.

At zero frequency, from Table XI, the numerical value of $\frac{S_m}{N}$ is 0.1540. Substituting these numerical values in equation (25),

B = 15,010 lines per sq. cm., approximately.

58. Electromotive Force Induced in Vibrator Strips. — The area swept over by a vibrator strip during a half-cycle may be determined approximately by assuming the strips to coincide with straight lines at the instant of maximum mirror deflection. From the known value of flux density, the total change of flux within the loop may be determined, and thus the induced e.m.f.

Suppose the vibrator to be carrying a current of 0.060 amp. and thus giving a deflection of 1.0 cm. on the screen. By equation (22), then, $S_m = \frac{dD}{2L} = 0.000166$ cm., and the total area enclosed by the two strips when each is displaced by this amount at the center is 0.000184 sq. cm. The flux enclosed by this loop is

$$\Phi_{max.} = 0.000184\, B = 2.76 \text{ lines.}$$

The effective value of induced e.m.f. is $4.44\, \Phi_{max.}\, f\, 10^{-8}$ volts.

$$E = 12.3 f\, 10^{-8} \text{ volts.}$$

Even for $f = 5000$ cycles per second, this gives only 0.00061 volts — an amount scarcely comparable with the minimum voltage impressed on the vibrator circuit. The resistance of the vibrator circuit may be reduced to approximately one ohm, thus requiring an e.m.f. of 0.060 volts to produce the amount of current under consideration. Thus under the most unfavorable conditions — highest frequency and lowest vibrator circuit resistance — the voltage induced in the vibrator strips by virtue of their motion in the magnetic field is only one per cent of the impressed, so that in no one case need any appreciable error be expected to arise from the cause here considered.

59. The Free Vibration Frequency of the System when the Inertia of the Mirror is Considered. — The vibrator mirror may oscillate freely at an infinite number of frequencies if the damping be not too great, and if the damping constant be zero these frequencies may be determined from equation (16) by equating the denominator in the parentheses to zero and then finding by trial the values of ω which satisfy the equation. This procedure amounts to finding the values of ω for which S_m will have a finite value even though there be no current flowing in the vibrator strips — that is, $N = 0$.

Thus, for free oscillation, undamped,

$$\frac{\omega^2}{H} \sin \alpha l - \alpha \cos \alpha l = 0. \qquad (26)$$

For $k = 0$, by equation (5),

$$\alpha^2 = \frac{m\omega^2}{T}, \quad \text{or} \quad \alpha = \omega\sqrt{\frac{m}{T}},$$

so that by replacing H by its equivalent, equation (11),

$$\frac{\omega^2 I}{T\,d^2}\sin\sqrt{\frac{m}{T}}\,l\omega - \sqrt{\frac{m}{T}}\,\omega\cos\sqrt{\frac{m}{T}}\,l\omega = 0, \tag{27}$$

from which, by dividing,

$$\left.\begin{aligned} \tan\sqrt{\frac{m}{T}}\,l\omega &= \frac{\sqrt{mT}\,d^2}{\omega I}, \\ \omega &= \frac{1}{l}\sqrt{\frac{T}{m}}\tan^{-1}\frac{\sqrt{mT}\,d^2}{\omega I}, \end{aligned}\right\} \tag{28}$$

either of which expressions may be used to determine ω.

A graphical method applied to the first of equations (28) yields sufficiently accurate results with very little labor. The two curves,

$$y' = \tan\sqrt{\frac{m}{T}}\,l\omega$$

and

$$y'' = \frac{\sqrt{mT}\,d^2}{I\omega},$$

may be plotted, using ω as independent variable, and their intersections noted. The values of ω at these intersections are those corresponding to free oscillations of the mirror.

Since the second curve, y'', is asymptotic to the axis of ω, the intersections of the two curves give values of ω more and more nearly equal to those for which

$$\tan\sqrt{\frac{m}{T}}\,l\omega = 0, \quad \text{or} \quad \omega = n\pi\sqrt{\frac{T}{ml^2}},$$

where n is any integer.

If no mirror be present the quantity I becomes zero in the equations, and then (28) reduces to

$$\left.\begin{aligned} \omega\Big]_{I=0} &= \frac{1}{l}\sqrt{\frac{T}{m}}\tan^{-1}(\infty), \\ \text{or} \qquad \omega\Big]_{I=0} &= (2n-1)\frac{\pi}{2l}\sqrt{\frac{T}{m}}, \end{aligned}\right\} \tag{29}$$

where n is any integer. For $n = 1$, the fundamental frequency of vibration for the strips alone becomes

$$f\Big]_{I=0} = \frac{\omega}{2\pi}\Big]_{I=0} = \frac{1}{4l}\sqrt{\frac{T}{m}}, \tag{30}$$

a result identical with that given in equation (19).

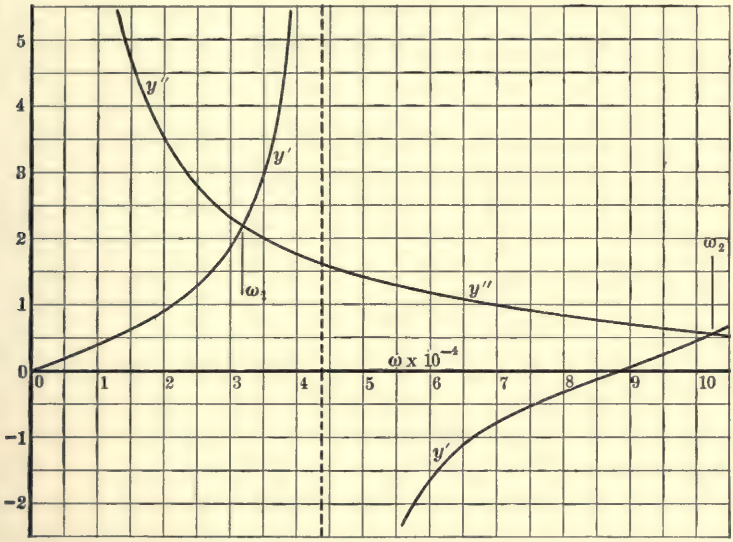

FIG. 21. — Determination of the Free Frequencies of Vibration of Oscillograph Mirror. $\omega_1 = 32{,}300$. $\omega_2 = 102{,}500$.

For the vibrator and mirror under consideration, equation (28) gives the numerical result,

$$\tan\,(0.000035415\,\omega) = \frac{70{,}808}{\omega}. \tag{31}$$

Fig. 21 shows the two curves,

$$y' = \tan\,(0.000035415\,\omega) \quad \text{and} \quad y'' = \frac{70{,}808}{\omega},$$

from the intersections of which the first two free vibration frequencies are

$$f_1 = \frac{32{,}300}{2\pi} = 5150; \quad f_2 = \frac{102{,}500}{2\pi} = 16{,}300.$$

The first of these values, 5150, may be compared with the free vibration frequency for the strips alone, 7050 cycles per second, as given by equation (19). The addition of the mirror to the system reduces the free vibration frequency by 1900 cycles per second.

It is to be noticed that equation (29) gives no free vibration frequency which is an even multiple of the fundamental frequency, obtained by placing $n = 1$, even though it is known that a stretched string or strip may oscillate at any frequency which is a multiple of the fundamental. It is to be remembered in this connection that equation (16), on which (29) is based, refers to the motion of the mid-point of the stretched strip only, and that for even multiples of the fundamental frequency this point would be at a node and therefore have no motion. For all odd multiples the mid-point lies at an anti-node, and thus has a motion — the result shown by (29).

CHAPTER VI

ALTERNATING CURRENT PHENOMENA IN AN IDEAL LINE

60. The Ideal Line. — By an ideal line, from the point of view taken in the preparation of this chapter, is meant one which is characterized by four electrical properties per unit of length, viz., line resistance, inductance, conductance, and electrostatic capacity. The line is supposed to be uniform throughout, so the values of these four constants or properties are constant for every unit of length. These constants must be regarded as the effective values of such, since the apparent properties of a transmission system carrying alternating currents are dependent upon the frequency of the line voltages and currents. A discussion of such changes will follow, for here it is sufficient to consider the constants as effective values and to develop the equations for the alternating phenomena in terms of them.

61. Fundamental Circuit Constants. — In general, for symmetrically arranged polyphase systems it is desirable to take the properties of each conductor as referred to the neutral of the system, although in the case of single phase transmission with two conductors it is satisfactory to consider the conductor with its return in determining the physical constants. In the first case, voltages are counted for each conductor to neutral, while in the second case, the voltage between wires is used.

Let

r = line resistance per unit of length, ohms.
g = line conductance per unit of length, mhos.
L = self-inductance per unit of length, henrys.
C = electrostatic capacity per unit of length, farads.

Since the numerical values of the line constants do not change (on the assumption of an ideal line) for a fixed frequency, a sine wave of voltage impressed on the line will produce only sine waves of voltage and current throughout as long as the load is not of such

a character as to cause a distortion of wave shape. This is in accordance with the discussion in section 48, Chapter IV, and permits of a representation of the line constants at this fixed frequency by means of the conventional complex quantity notation, impedance and admittance.

$$\left.\begin{aligned} Z &= r + j\,2\pi fL = \text{impedance per unit length.} \\ Y &= g + j\,2\pi fC = \text{admittance per unit length.} \\ x &= 2\pi fL, \qquad z = \sqrt{r^2 + x^2}, \\ b &= 2\pi fC, \qquad y = \sqrt{g^2 + b^2}. \end{aligned}\right\} \tag{1}$$

In the two quantities, $Z = r + jx$ and $Y = g + jb$, it is to be noted that the "j" terms are both of the same algebraic sign — a matter over which students very often become confused, since an admittance, Y, as the reciprocal of an impedance, Z, reverses the algebraic sign of the "j," or imaginary, term. In this instance, however, the admittance Y is not the reciprocal of, nor in any way related to, the impedance Z — the former being the conductance and capacity admittance from wire to neutral, or between wires, while the latter is the inductive impedance of the wire, or wires.

62. Fundamental Differential Equations and Solutions. — In the alternating current system the electrostatic capacity is always present, and therefore the admittance Y, so that in no case can it be said that the current is of uniform value throughout the line length — a condition approximately realized in the case of direct current transmission over a line with good insulation. For this reason the exact solution for the alternating current problem will be given first, and approximate solutions discussed afterwards, with the rigorous expressions on hand for comparison.

The fundamental differential equations for current and voltage along the line are set up in exactly the same way as for the direct current problem, equation (1), Chapter II, using Z and Y in place of r and g, respectively,

$$\frac{dI}{dl} = YE; \frac{dE}{dl} = ZI,$$

from which

$$\left.\begin{aligned} \frac{d^2I}{dl^2} - ZYI &= 0, \\ \frac{d^2E}{dl^2} - ZYE &= 0. \end{aligned}\right\} \tag{2}$$

Equation (2) is exactly similar to equation (3), Chapter II, the constant ZY taking the place of rg (to which ZY reduces for zero frequency), so that the complete solution may be written immediately, following equation (9), Chapter II.

$$\left.\begin{aligned} E &= \frac{1}{2}\left[\left(E_0 + I_0\sqrt{\frac{Z}{Y}}\right)\epsilon^{\sqrt{ZY}l} + \left(E_0 - I_0\sqrt{\frac{Z}{Y}}\right)\epsilon^{-\sqrt{ZY}l}\right], \\ I &= \frac{1}{2}\left[\left(I_0 + E_0\sqrt{\frac{Y}{Z}}\right)\epsilon^{\sqrt{ZY}l} + \left(I_0 - E_0\sqrt{\frac{Y}{Z}}\right)\epsilon^{-\sqrt{ZY}l}\right], \end{aligned}\right\} \tag{3}$$

where E_0 and I_0 are the voltage and current respectively at the load and E and I are the voltage and current respectively at a distance l from the load end. All of these four quantities are in general complex, since they represent electromotive forces or currents which are alternating.

63. Auxiliary Circuit Constants. — Since both Z and Y are complex, the two quantities $\sqrt{\frac{Z}{Y}}$ and $\sqrt{ZY}$ are, in general, both complex. Let

$$V = \sqrt{ZY} = \alpha + j\beta \tag{4}$$

serve as a defining equation for V, α, and β, which thereby become constants per unit of length of the transmission system. α and β may be explicitly expressed in terms of the fundamental line constants by the following method:

$$(\alpha + j\beta)^2 = (r + jx)(g + jb),$$

or

$$\alpha^2 + 2j\alpha\beta - \beta^2 = rg - xb + jgx + jrb,$$

from which by equating reals and imaginaries,

$$\left.\begin{aligned} \alpha^2 - \beta^2 &= rg - xb, \\ 2\alpha\beta &= gx + rb. \end{aligned}\right\} \tag{4a}$$

From these two equations, as Steinmetz shows (Transient Electric Phenomena and Oscillations),

$$\left.\begin{aligned} \alpha &= \sqrt{\tfrac{1}{2}(zy - xb + rg)}, \\ \beta &= \sqrt{\tfrac{1}{2}(zy + xb - rg)}. \end{aligned}\right\} \tag{5}$$

Though convenient for some purposes it is not desirable to use equation (5) for the calculation of numerical values of α and β, particularly when the line resistance r and leakage g are small in

comparison with x and b, respectively. In such an event the product zy differs very little from xb, and in the expression for α the resulting difference, $zy - xb$, would be difficult to obtain with accuracy. Besides, in any case the numerical work required is excessive. It is better to form

$$Z = r + jx = z\underline{/\theta_z}$$

$$Y = g + jb = y\underline{/\theta_y}$$

$$\sqrt{ZY} = \alpha + j\beta = V = \sqrt{zy}\underline{/\tfrac{1}{2}(\theta_z + \theta_y)} = v\underline{/\theta_v}$$

$$\alpha = v \cos \theta_v,$$

$$\beta = v \sin \theta_v,$$

and at the same time to form

$$\sqrt{\frac{Z}{Y}} = \sqrt{\frac{z}{y}}\underline{/\tfrac{1}{2}(\theta_z - \theta_y)}.$$

For convenience in writing, let

$$U = \sqrt{\frac{Z}{Y}} = u\underline{/\theta_u}. \tag{6}$$

64. Complete Expansion of Exact Solution. — If, now, the substitution

$$\epsilon^{\pm Vl} = \epsilon^{\pm(\alpha l + j\beta l)} = \epsilon^{\pm \alpha l}(\cos \beta l \pm j \sin \beta l) \tag{7}$$

be made in equation (3), we obtain

$$E = \tfrac{1}{2}(E_0 + UI_0)\,\epsilon^{\alpha l}(\cos \beta l + j \sin \beta l),$$
$$+ \tfrac{1}{2}(E_0 - UI_0)\,\epsilon^{-\alpha l}(\cos \beta l - j \sin \beta l)$$

$$I = \tfrac{1}{2}\left(I_0 + \frac{1}{U}E_0\right)\epsilon^{\alpha l}(\cos \beta l + j \sin \beta l)$$
$$+ \tfrac{1}{2}\left(I_0 - \frac{1}{U}E_0\right)\epsilon^{-\alpha l}(\cos \beta l - j \sin \beta l). \tag{8}$$

Equation (8) is particularly well adapted to the physical interpretation of the mathematical expressions, although, as in the solution of the problem for the transmission of direct currents over leaky lines, a more convenient form for computing purposes may be used.

65. Interpretation of the Various Terms in the Expanded Solution. — In equation (8) consider first the quantity

$$\cos \beta l \pm j \sin \beta l = \epsilon^{\pm j\beta l}.$$

The absolute value of this expression, as the square root of the sum of the squares of the two components, is alway unity, so it is evident that the only function such a factor can perform is to rotate, or change the phase position of, a vector representing an alternating quantity. In fact this quantity is called a "rotating operator," because it rotates any vector which is affected by it as a multiplier, through an angle βl.

$$\cos \beta l + j \sin \beta l = \epsilon^{+j\beta l}$$

rotates a vector in a counter-clockwise direction, and

$$\cos \beta l - j \sin \beta l = \epsilon^{-j\beta l}$$

rotates a vector in a clockwise direction — in both cases by an angle numerically equal to βl radians — while the length of the vector remains unchanged.

Returning to equation (8) it is seen that the line voltage (the same remarks apply to the line current) consists of the vector sum of two apparently distinct components. For increasing values of l, that is, going from load towards generator, the vector representing the first component,

$$E' = \tfrac{1}{2}(E_0 + UI_0)\,\epsilon^{\alpha l}(\cos \beta l + j \sin \beta l), \tag{9}$$

rotates in a counter-clockwise direction by an amount proportional to l, and at the same time increases in length on account of the factor $\epsilon^{\alpha l}$. Counter-clockwise rotation of a vector signifies an advance in phase position of the harmonically varying quantity represented conventionally by this vector. Since the variation of e' with time is simple harmonic at a fixed point in the line, and for different points along the line the phase position of e' advances proportionally to the distance, it appears that this quantity E' is merely the vector representation of a voltage wave moving in a negative direction along the line (generator towards load) and decreasing in magnitude as it moves along. Since E' at points near the source of power is advanced in phase position with respect to E' at points more remote, it is obvious that the direction of the wave motion of this component is opposite to our arbitrarily selected positive direction along the line. For increase in l, this component increases, but speaking of the wave represented by E', it may be said to decrease in magnitude in the direction of its propagation.

As to the remaining component,

$$E'' = \tfrac{1}{2}(E_0 - UI_0)\,\epsilon^{-\alpha l}(\cos\beta l - j\sin\beta l), \tag{10}$$

it is seen that the vector which this represents rotates in a clockwise direction as l increases, and at the same time decreases in magnitude according to the multiplier $\epsilon^{-\alpha l}$. E'' is thus the vector representation of a wave moving along the line from the load towards the generator — decreasing in value as it goes. According to the above physical interpretation of the meaning of the two terms in the expression for the line voltage, the e.m.f. at any point is made up of the sum of two separate waves moving in opposite directions; a main wave, E', moving from generator towards load, and a reflected wave, E'', moving from the load towards the generator. Both of these waves decrease in magnitude at the same proportionate rate, in the direction of their propagation. This decrease in amplitude of each of the separate waves is due to the loss of energy in the resistance and leakage of the line, that is, the i^2r and e^2g losses. For a line with no resistance or leakage the waves do not change at all in magnitude, though the effective line voltage or current of course varies along the line on account of the different phase positions at which the main and reflected waves combine into the resultant.

As stated before, the current equation may be interpreted in exactly the same manner as has been done with the expression for voltage. The two components of the resultant current are

$$I' = \frac{1}{2}\left(I_0 + \frac{1}{U}E_0\right)\epsilon^{\alpha l}(\cos\beta l + j\sin\beta l), \tag{10a}$$

which represents the main wave, or wave traveling from generator towards receiver, and

$$I'' = \frac{1}{2}\left(I_0 - \frac{1}{U}E_0\right)\epsilon^{-\alpha l}(\cos\beta l - j\sin\beta l), \tag{10b}$$

which represents the reflected wave, traveling in the opposite direction from the main wave.

66. Example of Calculation of Fundamental Constants in the Equations. — Before going further with a discussion of the equations, a numerical example will be considered in order to illustrate the method of determining the numerical values of the quantities which enter into the equations and to form a basis for the construction of the vector diagrams representing the phenomena.

TABLE XII

Calculation of Auxiliary Constants of Transmission Circuit

Fundamental line constants, per mile to neutral:

$r = 0.275$, $g = 0.15 \times 10^{-6}$, $L = 0.00204$, $C = 0.0146 \times 10^{-6}$,
f = frequency = 60 cycles, Log L = 7.309630–10, Log C = 2.164353–10.

	$\log f$	1.778151	
	$\log 2\pi f$	2.576331	
$x = 2\pi f L$	$\log x$	9.885961–10	
	$\log r$	9.439333–10	
$\tan\theta_z = \dfrac{x}{r}$	$\log\tan\theta_z$	0.446628	
	$\log\sin\theta_z$	9.973872–10	
$z = \dfrac{x}{\sin\theta_z}$	$\log z$	9.912089–10	$Z = r + jx = z\underline{/\theta_z}$
	θ_z	70° 19′ 26.42″	
$b = 2\pi f C$	$\log b$	4.740684–10	
	$\log g$	3.176091–10	
$\tan\theta_y = \dfrac{b}{g}$	$\log\tan\theta_y$	1.564593	
	$\log\sin\theta_y$	9.999839–10	
$y = \dfrac{b}{\sin\theta_y}$	$\log y$	4.740845–10	$Y = g + jb = y\underline{/\theta_y}$
	θ_y	88° 26′ 20.16″	
$v^2 = zy$	$2\log v$	4.652934–10	$V^2 = ZY = zy\underline{/\theta_z + \theta_y}$
$2\theta_v = \theta_z + \theta_y$	$2\theta_v$	158° 45′ 46.58″	
	θ_v	79° 22′ 53.29″	
	$\log v$	7.326467–10	$V = \alpha + j\beta = v\underline{/\theta_v}$
	$\log\cos\theta_v$	9.265453–10	
	$\log\sin\theta_v$	9.992498–10	
$\alpha = v\cos\theta_v$	$\log\alpha$	6.591920–10	
$\beta = v\sin\theta_v$	$\log\beta$	7.318965–10	
	$\log\dfrac{360}{2\pi}$	1.758123	
$\beta° = \dfrac{360}{2\pi}\beta$	$\log\beta°$	9.077088–10	
	α	0.000390769	
	$\beta°$	0.119423	
$u^2 = \dfrac{z}{y}$	$2\log u$	5.171244	$U = \sqrt{\dfrac{Z}{Y}} = u\underline{/\theta_u}$
$2\theta_u = \theta_z - \theta_y$	$2\theta_u$	−(18° 06′ 53.74″)	
	$\log u$	2.585622	
	θ_u	−(9° 03′ 26.87″)	
	$\log\dfrac{1}{u}$	7.414378–10	$\dfrac{1}{U} = \sqrt{\dfrac{Y}{Z}} = \dfrac{1}{u}\underline{/-\theta_u}$
	U	385.143 $\underline{/-(9° 03′ 26.87″)}$	
	$\dfrac{1}{U}$	0.00259644$\underline{/9° 03′ 26.87″}$	

The numerical calculation of the line constants is given in detail, and, as is most convenient, is carried out by means of logarithms. Even when carried out with the accuracy afforded by six-place logarithm tables the amount of work required is very small, no work being required other than that indicated in the table. In general it would not be advisable to make such a computation with six-place tables, since the accuracy of the computation is very much greater than that of the fundamental data. Five places are usually sufficient.

67. Vector Diagrams Representing the Exponential Form of Equations. — Suppose that at the load the e.m.f. between one wire and neutral is 50,000 volts, and that the load current is 25.0 amperes. Suppose the power-factor of the load to be such as to cause the current to lag 25.0 degrees behind the voltage in time-phase position. Using the numerical values of the constants as determined from the calculations in Table XII, with equation

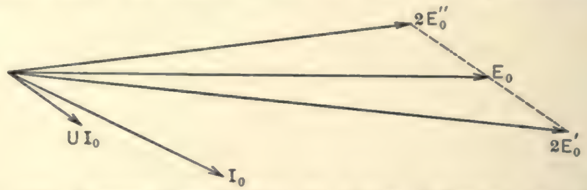

FIG. 22. — The Vectors Representing the Main, Reflected, and Resultant Voltage Waves at the Load End.

(8) the vector diagrams representing the separate components, as well as the resultant, of the voltage and current along the line will be constructed.

If the load voltage be selected as the reference vector (that is, the arbitrary origin of time so selected as to make E_0 a cosine variation), we have

$$E_0 = 50{,}000\underline{/0^\circ} \text{ volts,}$$

$$I_0 = 25.0\underline{/-25^\circ} \text{ amperes,}$$

and thus

$$UI_0 = 9628.58\underline{/-(34^\circ\ 03'\ 26''.87)} \text{ volts,}$$

$$\frac{1}{U} E_0 = 129.822\underline{/9^\circ\ 03'\ 26''.87} \text{ amperes.}$$

Fig. 22 shows the vectors, E_0, I_0, UI_0, and the two vectors representing the component waves at the receiving end,

$$2\,E_0' = E_0 + UI_0 \qquad \text{and} \qquad 2\,E_0'' = E_0 - UI_0.$$

Fig. 23 shows the analogous current vectors, with E_0 again shown as the reference, E_0, I_0, $\frac{1}{U} E_0$, and

$$2\,I_0' = I_0 + \frac{1}{U}\,E_0 \qquad \text{and} \qquad 2\,I_0'' = I_0 - \frac{1}{U}\,E_0.$$

If now a numerical value be assigned to l, say 100 (miles), the vector $2\,E_0'$ will be rotated in a counter-clockwise direction

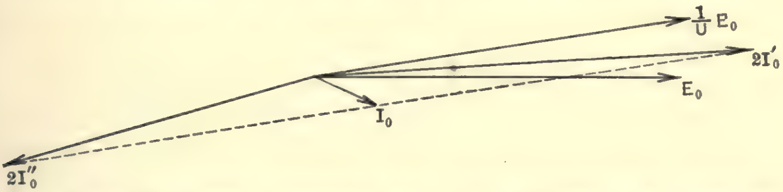

FIG. 23. — The Vectors Representing the Main, Reflected, and Resultant Current Waves at the Load End.

through an angle βl, or in this case 11.9423 degrees, and at the same time increased in length by the factor

$$\epsilon^{\alpha l} = \epsilon^{0.0390769} = 1.03985.$$

$2\,E_0''$ rotates by an equal amount in a clockwise direction, and decreases in length by the factor

$$\epsilon^{-\alpha l} = 0.96168.$$

One-half the vector sum of these two vectors is the vector representing the actual line voltage at the distance of one hundred miles from the load. This resultant is very easily obtained by drawing a vector from the origin to the mid-point of the line joining the extremities of the component vectors.

In Fig. 24, E_0 is the load voltage, to scale; $2\,E_0'$ and $2\,E_0''$ are twice the load-end values of the main and reflected waves, respectively. The curve, Locus of $2\,E'$, indicates the path followed by the end of the vector in question for continuously increasing values of l. For each 400 mile point in the line, the vector is drawn in the diagram. The last vector so drawn is for a line 2000 miles in length. For $l = 2000$ the rotation of $2\,E'$ is 238.846 degrees, and the factor $\epsilon^{\alpha l}$ is 2.1848. The curve, Locus of $2\,E''$, indicates the path followed by the end of the vector $2\,E''$ for continuously increasing line lengths. The spiral is in this case a decreasing one, and for $l = 2000$, the factor $\epsilon^{-\alpha l} = \dfrac{1}{2.1848} = 0.45770.$

The resultant of the two waves, E' and E'' is given by the vector E, the end of which moves along the curve, Locus of E.

The rotation of the vector E is not proportional to the line length as is the case with the separate components, since it is

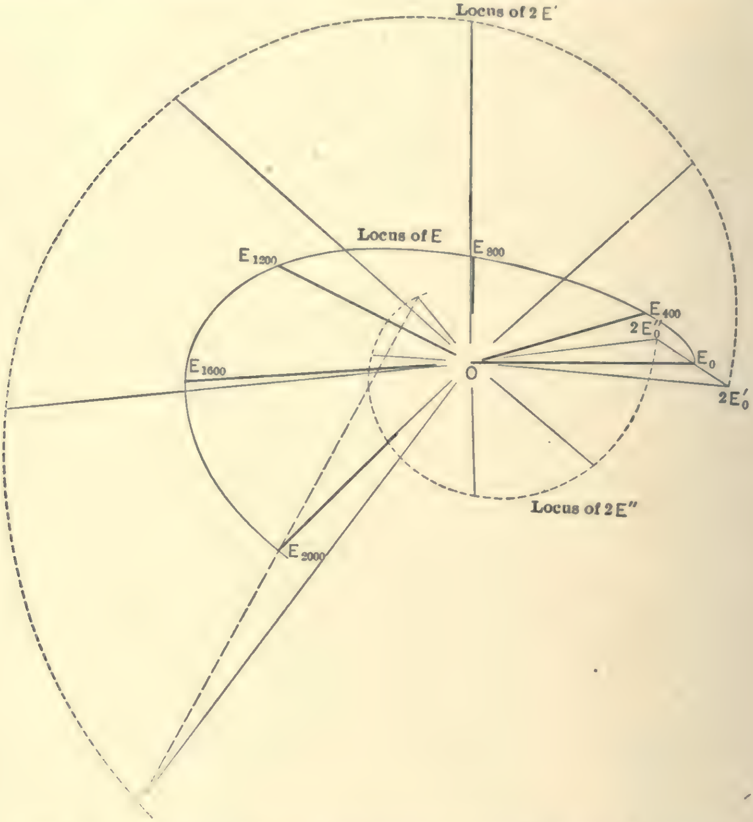

FIG. 24. — Diagram of Voltage Vectors for Increasing Line Length. Subscripts Refer to Line Length.

made up of the vector sum of two vectors rotating in opposite directions and changing in length as they rotate.

Fig. 25 contains the current vectors, all of which are treated in the way just described for the voltage vectors. For the load selected as illustration, the e.m.f. UI_0, which is added to and subtracted from the e.m.f. E_0 in order to form the initial vectors

for the separate waves, is considerably smaller than E_0, and therefore the initial vectors E_0' and E_0'' do not differ very much from E_0. Thus, for distances up to approximately 100 miles from the load end, the resultant voltage vector does not change very much.

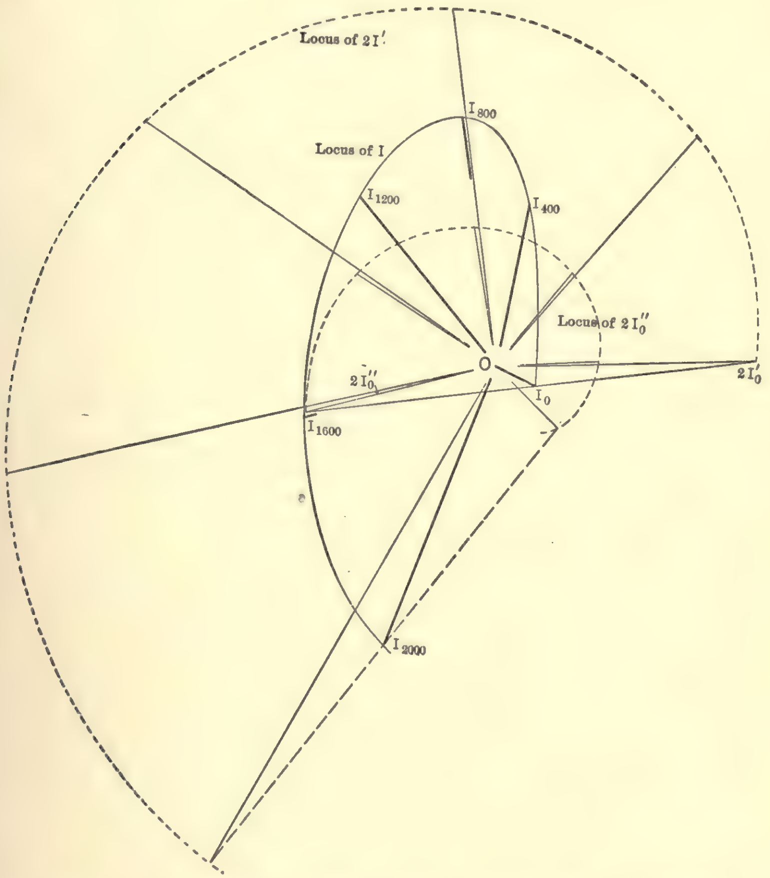

Fig. 25. — Diagram of Current Vectors for Increasing Line Length. Subscripts Refer to Line Length.

In the case of the vector diagram of currents, the component $\frac{1}{U} E_0$ is several times the load current, so that the current at the receiving end, I_0, is made up of the sum of two vectors nearly

equal in length and displaced from each other by approximately 180 degrees. The rotation in opposite directions of these two component vectors produces a resultant vector I which varies very rapidly for small increase in line length, as may be noticed in Fig. 25 from the manner in which the curve, Locus of I, starts from its initial point at I_0.

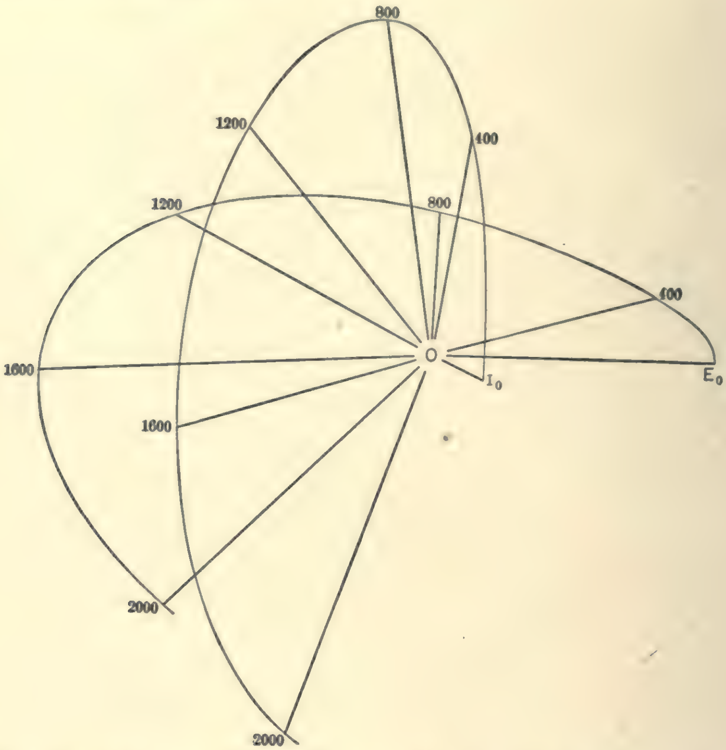

Fig. 26. — Voltage and Current Vectors. Numbers Indicate Line Length.

The two loci of the ends of the resultant voltage and current vectors are reproduced in Fig. 26, the vectors for the end of each 400 mile section of line being drawn. For any point in the line, the power-factor is given by the cosine of the angle between the vectors representing E and I for this length. For example, in Fig. 26, where the numbering indicates the length of line to which each vector belongs, for the end of the 800 mile section the current

is ahead of the voltage in time-phase position by an angle of approximately 10.5 degrees, while at a point 1200 miles distant from the load the current is lagging by 22.8 degrees. Again, at 2000 miles, the current leads by 25.8 degrees. For increasing line length this power-factor angle oscillates about an angle equal to that of the quantity $U = \sqrt{\frac{Z}{Y}}$, as will be shown later. Since the load as well as the line losses must be supplied by an average flow of power along the line in a negative direction (generator towards load) it is evident that this phase angle can never be greater than 90 degrees, for then the average power transmitted ($EI \cos \phi$) would reverse in sign, and thereby indicate a flow of energy in the opposite direction. In fact, for any line with losses (r or g present) the power-factor angle can never equal 90 degrees, for then there would be no average power passing the point in question to supply the line losses in the portion of the line more distant from the generator.

68. Operating Characteristics from Vector Diagrams. — From the original drawings for the preceding figures, in which the vectors were inserted for every 200 mile section, the following values were obtained:

TABLE XIII

DISTRIBUTION OF ELECTRICAL QUANTITIES ALONG A LONG TRANSMISSION LINE

Length, miles	$E_{eff.}$, volts	$I_{eff.}$, amp.	Power-factor angle, degrees	K.W. per phase	Effic., per cent
0	50,000	25.0	25.0 lag	1130	100.0
200	48,900	49.5	58.5 lead	1245	90.9
400	40,500	94.5	65.0 lead	1612	70.1
600	28,900	128.0	49.8 lead	2385	47.3
800	24,300	143.2	10.5 lead	3410	33.2
1000	34,700	139.5	19.5 lag	4550	24.8
1200	49,200	122.8	22.8 lag	5550	20.4
1400	60,600	107.0	8.5 lag	6400	17.7
1600	65,700	112.5	13.2 lead	7250	15.6
1800	65,000	140.0	26.2 lead	8150	13.9
2000	61,700	173.5	25.8 lead	9620	11.7

(The above values were determined by measuring the vector diagram, so may be slightly inaccurate.)

The above numerical results are plotted in Fig. 27, with line length as independent variable.

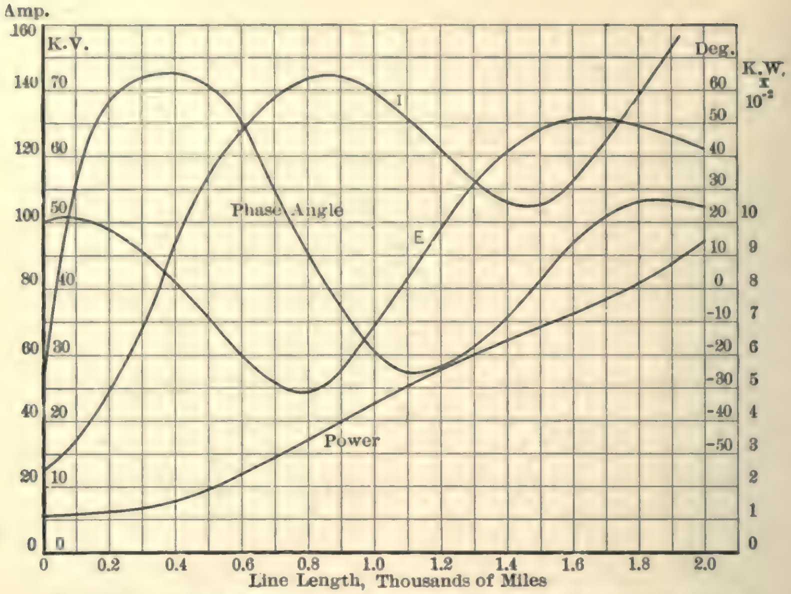

FIG. 27. — Voltage, Current, Power, and Phase Angle in a Long Line.

69. Limiting Forms of Vector Diagrams for Infinite Length Line. — In this figure the curves representing E and I approach the simple exponential curve in form when the line length becomes infinite. This follows from the disappearance of the term representing the reflected wave, in equation (8), because of the decreasing exponential. The equations of the limit-curves are thus

$$E = E_0'\epsilon^{\alpha l},$$

and

$$I = I_0'\epsilon^{\alpha l},$$

in which scalar values are used. The wavy appearance of the curves in Fig. 27 for the line under consideration is due to the combination of the main wave with the reflected wave, which, for the comparatively short length of line, has not disappeared.

The power curve, on the other hand, does not approach a simple exponential curve as a limit, as will be shown under the discussion of power relations and line losses.

70. Instantaneous Values from Vector Diagrams. — If it is desired to know the instantaneous distribution of current and e.m.f. along the line it is, of course, only necessary to draw the circles on the vectors in the diagram as diameters, and then to

measure the intercept on the time radius drawn through the origin at an angle with the reference vector equal to the product of the angular velocity of alternation ($\omega = 2\pi f$) and the elapsed time between the zero instant and the instant for which the distribution is desired. If the diagram represents effective values, the intercepts must be multiplied by the square root of two in order to secure instantaneous values.

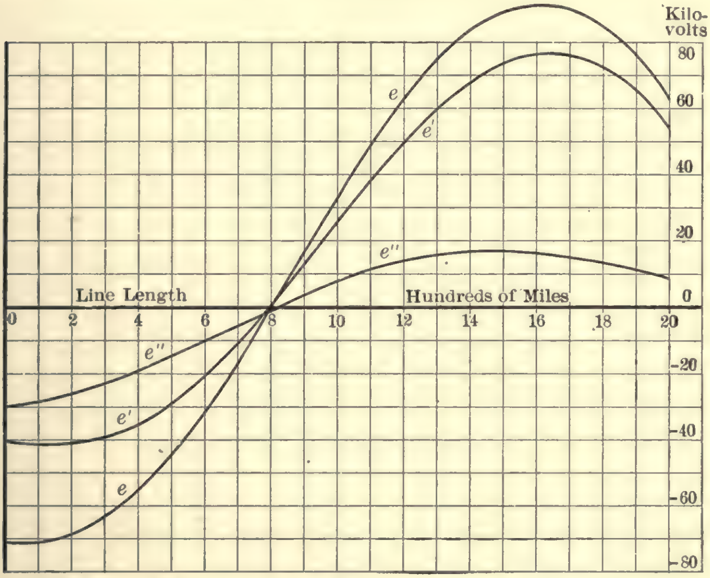

FIG. 28. — Instantaneous Distribution of Voltage 0.075 Second After Positive Maximum of E_0.

By selecting E_0 as the reference vector in this numerical illustration, we have, as functions of time,

$$e_0 = \sqrt{2}\, 50{,}000 \cos 2\pi ft,$$

referred to the instant at which e_0 passes through its positive maximum, and

$$i_0 = \sqrt{2}\, 25.0 \cos (2\pi ft - 25°).$$

Let the curves be plotted showing the instantaneous distribution of current and voltage 0.075 second after the origin of time. The time radius will have turned through $0.075 \times 60 = 4.50$ complete revolutions, or will be displaced from the reference line by an angle

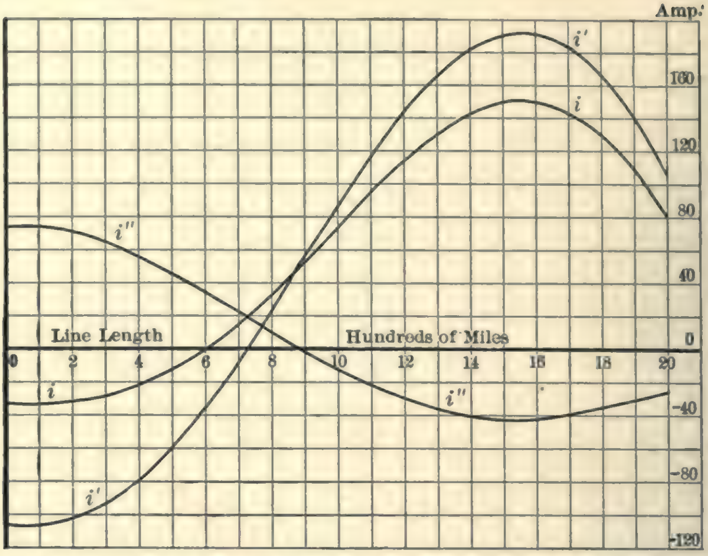

FIG. 29. — Instantaneous Distribution of Current 0.075 Second After Positive Maximum of E_0.

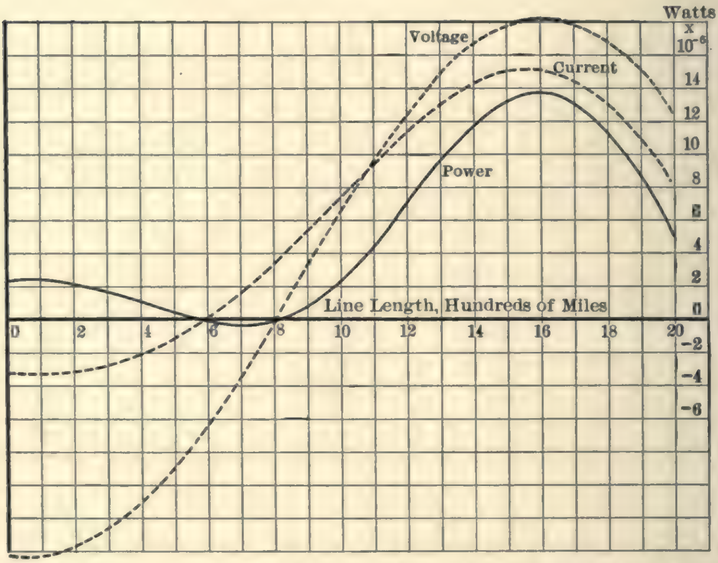

FIG. 30. — Instantaneous Distribution of Power 0.075 Second After Positive Maximum of E_0. Voltage and Current Curves Reproduced from Figs. 28 and 29.

of 180 degrees. The square root of two times the intercepts of the various circles on this radius gives the values which are plotted in Figs. 28, 29, and 30.

71. Instantaneous Values of Power at Different Points. — In Fig. 30, the instantaneous values of power are obtained from the products of the corresponding instantaneous values of current and e.m.f. It is interesting to note that at this particular instant, for the section of line between $l = 600$ and $l = 800$ miles, the power is negative — that is, the flow of energy is from load towards generator — while on either side of this section the energy flow is from generator towards load. At any point in the line at which the power-factor is not unity, these negative values of instantaneous power may be observed at some instant. From Fig. 27 it is seen that at the distances of 40, 870, and 1480 miles from the load the angle of lag is zero (power-factor unity), and thus the instantaneous power at these points cannot be negative at any time. These large variations in the distribution of instantaneous power along the line are of course occasioned by the continual and periodic redistribution of the energy stored in the line self-inductance and electrostatic capacity.

To show the variation in power distribution along the line from instant to instant, Fig. 31 contains curves giving the instantaneous values of power for six successive intervals of time, beginning with the instant at which E_0 passes through its positive maximum ($t = 0$) and covering one-half of a cycle.

No. 0, $t = 0$.
No. 1, $t = 1/720$ second, 1/12 cycle.
No. 2, $t = 2/720$ second, 2/12 cycle.
No. 3, $t = 3/720$ second, 3/12 cycle.
No. 4, $t = 4/720$ second, 4/12 cycle.
No. 5, $t = 5/720$ second, 5/12 cycle.

Though this set of curves covers only a half-cycle of voltage or current, it is sufficient to cover a whole cycle of the power wave, since this latter is of double the frequency of its two components, voltage and current.

72. Equations for the Loci of the Ends of the Component Vectors. — The curves traced out by the ends of the vectors which represent the component waves are logarithmic, or equiangular, spirals. Take, for example, the locus of the end of the

vector representing the main wave. Its initial length is e_0', and for increasing distance from the load end of the line, its length is represented by the equation

$$e' = e_0' \epsilon^{al}, \tag{11}$$

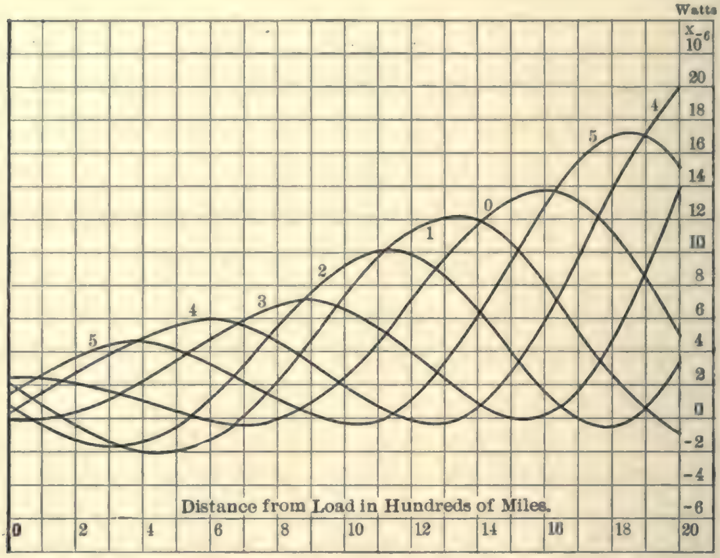

FIG. 31. — Instantaneous Distribution of Power 0, 1, 2, 3, 4, and 5 Twelfths of a Cycle after the Positive Maximum of E_0. Numbers on Curves Indicate Twelfths of a Cycle.

and the angle the vector makes with its initial position is

$$\theta = \beta l. \tag{12}$$

Eliminating the parameter l by $l = \dfrac{\theta}{\beta}$,

$$e' = e_0' \epsilon^{\frac{\alpha}{\beta}\theta}, \tag{13}$$

which is the equation of the exponential curve in polar coördinates.

Let ϕ = the angle between the tangent to the curve and the radius vector. Then from the geometry of the figure,

$$\frac{de'}{d\theta} = e' \cot \phi. \tag{14}$$

By differentiating (13),

$$\frac{de'}{d\theta} = e_0' \frac{\alpha}{\beta} e^{\frac{\alpha}{\beta}\theta} = \frac{\alpha}{\beta} e', \tag{15}$$

so that by comparing equations (14) and (15), there results

$$\cot \phi = \frac{\alpha}{\beta}. \tag{16}$$

From this it is seen that cot ϕ, and therefore ϕ, is a constant. It is from this property that the curve derives the appropriate name of "Equiangular Spiral."

In the expression

$$V = \sqrt{ZY} = v\underline{/\theta_v},$$
$$\theta_v = \tfrac{1}{2}(\theta_z + \theta_y),$$

and, as previously shown,

$$\alpha = v \cos \theta_v,$$
$$\beta = v \sin \theta_v,$$

so that

$$\frac{\alpha}{\beta} = \cot \theta_v.$$

Thus, from (16), the angle ϕ between the tangent to the curve and the radius vector is the same as the angle θ_v, which, in turn, is equal to the mean of the impedance and admittance angles.

$$\phi = \theta_v = \tfrac{1}{2}(\theta_z + \theta_y). \tag{17}$$

For the decreasing spiral, the angle ϕ is the same as above — the rotation of the radius vector in the opposite direction causing a diminution instead of an increase in the length of the radius.

73. Mechanical Construction of Equiangular Spirals. — The equiangular property of these curves suggests the possibility of constructing them by some purely mechanical means. This may be done by the use of the apparatus shown in Fig. 32, in which a small sharp-edged tracing wheel is mounted in a frame so that the plane of the wheel may be set at any desired angle ϕ with the axis of the instrument. Along the axis of the instrument is a narrow slit, which slit slides over a pin placed at the center of the vector diagram, at 0. A scale of degrees and a vernier may be placed on the rotatable mounting of the tracing wheel so that the angle ϕ may be set off accurately. The zero of the scale may be located by

ascertaining by trial the position at which the wheel is to be set so as to trace out a circle. By pressing the wheel firmly against the paper and rotating the entire apparatus about the point 0, an equiangular spiral will be traced out as shown — the angle ϕ at which the instrument is set being the same as the angle ϕ in equations (16) and (17). As shown, rotation in a counter-clockwise

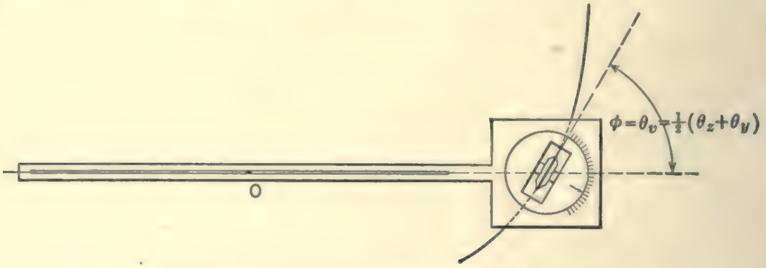

FIG. 32. — Polar Exponential Curve Tracer.

direction gives the increasing spiral, and rotation in the opposite direction, the decreasing spiral.

Using such a device, it is only necessary to draw in the diagram the two initial vectors, $2\,E_0'$ and $2\,E_0''$, and with the angle ϕ set off on the vernier and scale, to place the tracing wheel upon the end of each vector and to draw in the spirals. This done, the vectors, $2\,E'$ and $2\,E''$, corresponding to any particular line length are obtained by drawing the lines which make angles βl with the initial vectors, from the center of the diagram to the curves. The resultant vector is obtained as before, by drawing the line from the origin to the mid-point of the line joining the extremities of the two component vectors, $2\,E'$ and $2\,E''$.

By the use of such a device, comparatively accurate solutions for lines of different length, but with constant receiver load, may be obtained very rapidly after having at first computed the numerical values of the auxiliary constants, α, β, and U, as per Table XII.

74. Introduction of Hyperbolic Functions of Complex Variables. — In the preceding section the manner of the variation of the line phenomena with change in line length has been discussed. The vector diagrams, for any distance from the receiving end of the line have been shown to be very easily obtained by the use of a mechanical device for tracing the exponential curves.

Such procedures are particularly applicable to cases of constant receiver load and variable line length.

In dealing with a line of given length over which a variable load is to be supplied, a more convenient form of expression than the exponential in equation (8), as exemplified in the vector diagrams just given, may be obtained by using hyperbolic functions.

By combining terms in equation (3), the complete solution may be put into the form

$$\left.\begin{aligned} E &= E_0 \frac{1}{2}\left(\epsilon^{\sqrt{ZY}l} + \epsilon^{-\sqrt{ZY}l}\right) + I_0 \sqrt{\frac{Z}{Y}}\,\frac{1}{2}\left(\epsilon^{\sqrt{ZY}l} - \epsilon^{-\sqrt{ZY}l}\right), \\ I &= I_0 \frac{1}{2}\left(\epsilon^{\sqrt{ZY}l} + \epsilon^{-\sqrt{ZY}l}\right) + E_0 \sqrt{\frac{Y}{Z}}\,\frac{1}{2}\left(\epsilon^{\sqrt{ZY}l} - \epsilon^{-\sqrt{ZY}l}\right), \end{aligned}\right\} \quad (18)$$

which, by introducing

$$V = \sqrt{ZY}, \qquad \text{and} \qquad U = \sqrt{\frac{Z}{Y}},$$

and the hyperbolic functions for the combinations of exponentials, become

$$\left.\begin{aligned} E &= E_0 \cosh Vl + I_0\, U \sinh Vl, \\ I &= I_0 \cosh Vl + E_0 \frac{1}{U} \sinh Vl. \end{aligned}\right\} \quad (19)$$

Naturally, equation (19) is identical in form with equation (14) of Chapter II for the direct current system. $V = \sqrt{ZY}$ is of course complex, so that the hyperbolic functions of a complex quantity are required.

$$V = \alpha + j\beta.$$

$$\left.\begin{aligned} \cosh Vl &= \cosh \alpha l \cos \beta l + j \sinh \alpha l \sin \beta l, \\ \sinh Vl &= \sinh \alpha l \cos \beta l + j \cosh \alpha l \sin \beta l. \end{aligned}\right\} \quad (20)$$

Equation (19) for the alternating current problem may be carried through all the developments as are given in Chapter II for the direct current system, keeping in mind, of course, the fact that the complex quantity notation must be retained.

75. Calculation of Constants in Equations for a Line of Given Length. — The utmost convenience and dispatch are to be found in the use of equations (19) for the determination of the operating characteristics of a transmission system, and particularly if a combination of analytical and graphical methods is employed. A numerical example will serve best to indicate the method.

Let it be required to plot curves showing the operating characteristics of a transmission line whose constants are those given in Table XII and whose length is 400 miles, for the condition of constant voltage at the load end.

The first step in the work is to obtain the numerical values of the three coefficients

$$\cosh Vl, \quad U \sinh Vl, \quad \text{and} \quad \frac{1}{U}\sinh Vl,$$

as required in equation (19). From the values of α and β given in Table XII and using tables of the logarithms of hyperbolic functions, the computations may be conveniently carried out as shown below in Table XIV.

TABLE XIV.

COMPUTATION OF CONSTANTS IN HYPERBOLIC EXPRESSIONS FOR TRANSMISSION LINE PHENOMENA

Line length = 400 miles

From Table XII,	α =	0.000390769	
	β	0.119423 degrees	
	αl	0.156308	
	βl	47.769 degrees	
	$\log \cosh \alpha l$	0.00528	
	$\log \sinh \alpha l$	9.19575–10	
	$\log \sin \beta l$	9.86949–10	
	$\log \cos \beta l$	9.82745–10	
	$\log A_c = \log \cosh \alpha l \cos \beta l$	9.83273–10	$\cosh Vl = A_c + jB_c$
	$\log B_c = \log \sinh \alpha l \sin \beta l$	9.06524–10	
	$\log \tan \theta_c$	9.23251–10	
	$\log \cos \theta_c$	9.99376–10	
	$\log \cosh Vl$	(9.83897–10) $\underline{/9.693 \text{ degrees}}$	
	$\log A_s = \log \sinh \alpha l \cos \beta l$	9.02320–10	$\sinh Vl = A_s + jB_s$
	$\log B_s = \log \cosh \alpha l \sin \beta l$	9.87477–10	
	$\log \tan \theta_s$	0.85157	
	$\log \sin \theta_s$	9.99574–10	
	$\log \sinh Vl$	(9.87903–10) $\underline{/81.989 \text{ degrees}}$	
From Table XII,	$\log U$	(2.58562) $\underline{/-9.057 \text{ degrees}}$	
	$\log U \sinh Vl$	(2.46465) $\underline{/72.932 \text{ degrees}}$	
	$\log \frac{1}{U} \sinh Vl$	(7.29341–10) $\underline{/91.046 \text{ degrees}}$	
Results of Computation	$\cosh Vl$	0.69020 $\underline{/9.693 \text{ degrees}}$	
	$U \sinh Vl$	291.51 $\underline{/72.932 \text{ degrees}}$	ohms.
	$\frac{1}{U} \sinh Vl$	0.0019652 $\underline{/91.046 \text{ degrees}}$	mhos.

Thus, numerically,

$$E = E_0 [0.69020\underline{/9^\circ.693}] + I_0 [291.51\underline{/72^\circ.932}]$$

$$I = I_0 [0.69020\underline{/9^\circ.693}] + E_0 [0.0019652\underline{/91^\circ.046}]$$

76. Graphical Treatment for Various Load-end Conditions. — From this point on, the remainder of the solution may be obtained graphically by constructing the vector diagrams representing equation (19) on the basis of the numerical values furnished by Table XIV. Since the load voltage is supposed to be kept constant, the vector diagram is constructed on E_0 as reference

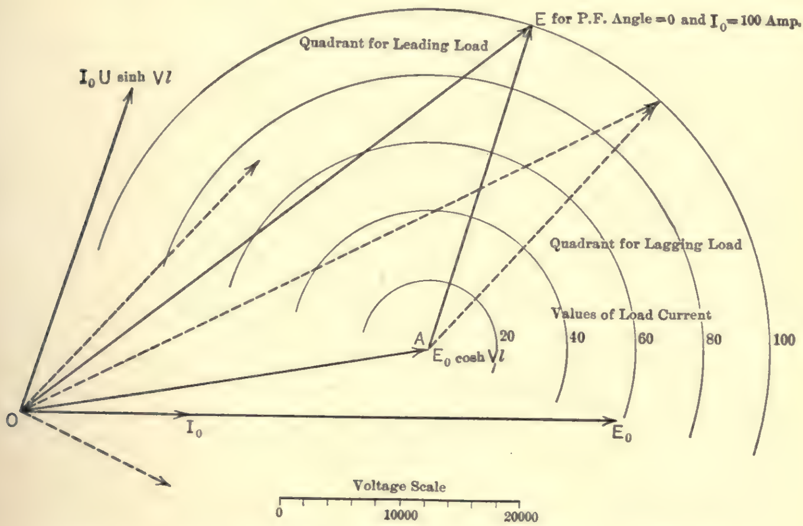

FIG. 33. — Voltage Diagram for Various Load Currents. Based on Equation (19).

vector. Let E_0 = 50,000 volts, effective. Fig. 33 is the voltage diagram, representing the equation

$$E = E_0 \cosh Vl + I_0 U \sinh Vl.$$

The resultant voltage at the generator is made up of two parts — one part proportional to the load voltage and the other to the load current. The vector representing $E_0 \cosh Vl = 34{,}510\underline{/9.693 \text{ deg.}}$ is drawn at an angle of 9.693 degrees from E_0 and with a length to the same scale as E_0 representing 34,510 volts. This vector, which is one of the two which go to make up the resultant vector for E, remains fixed regardless of the load current I_0. In the diagram, A designates the end of this vector.

To complete the diagram and make it applicable to any and all

values of load current at any power-factor, select a base value of current of, say, 100 amperes at unity power-factor. Then

$$I_0 = 100\underline{/0\text{ deg.}},$$

and $$I_0 U \sinh Vl = 29{,}151\underline{/72.932\text{ deg.}}\text{ volts.}$$

The resultant generator voltage at E is obtained by adding the two vectors in the ordinary manner, giving, for this particular load, $E = 54{,}300\underline{/38.2\text{ deg.}}$ by measurement of the diagram. This gives the solution for only one value of load current, but it is readily seen that for any other value of load current at this power-factor, the solution is obtained by taking the proportional value of the vector $I_0U \sinh Vl$, and, of course, of AE. For example, the line AE may be divided into five equal parts, giving thereby the values of generator voltage for $I_0 = 0$, 20, 40, 60, 80, and 100 amperes. An extension of the vector beyond the length of AE gives solutions for load currents greater than 100 amperes.

The length of the vector $I_0U \sinh Vl$ depends only upon the absolute value of I_0, while the angle it makes with the reference line OE_0 depends only upon the power-factor of the load, and is, in fact, equal to the sum of the power-factor angle and the angle of the factor $U \sinh Vl$. Thus, for any power-factor of load, it is only necessary to turn the vector $I_0U \sinh Vl$, as drawn in the diagram for unity power-factor, through an angle equal to the load power-factor angle. Rotation of this vector also rotates the line AE through an equal angle. The dotted lines in the figure indicate the vectors representing the solution for a load current of 100 amperes lagging 25 deg. behind the load voltage. Arcs of circles drawn through the points subdividing the line AE provide solutions for the intermediate values of current at any power-factor.

To obtain the value of the generator current we again take the sum of two components — one proportional to E_0, and therefore constant in this particular case, and the other proportional to I_0. Fig. 34 is the diagram of current vectors with E_0 again used as reference. The quantity $E_0 \frac{1}{U} \sinh Vl$ required by equation (19) is equal to

$$(50{,}000\underline{/0\text{ deg.}})\ (0.0019652\underline{/91.046\text{ deg.}})$$
$$= 98.26\underline{/91.046\text{ deg.}}\text{ amperes.}$$

This current is represented by the vector OA. To it is to be added the remaining component $I_0 \cosh Vl$ which, in this case, depends upon the load current and the power-factor. Select a base value

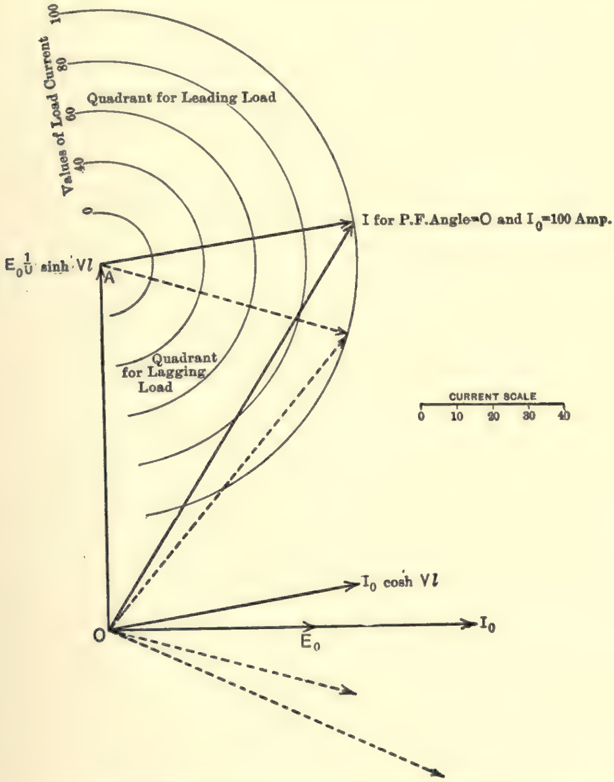

FIG. 34. — Current Diagram for Various Load Currents. Based on Equation (19).

of current of one hundred amperes at unity power-factor as was done in constructing the voltage diagram. Then

$$I_0 \cosh Vl = (100\underline{/0\text{ deg.}})\ (0.69020\underline{/9.693\text{ deg.}})$$
$$= 69.02\underline{/9.693\text{ deg.}}\text{ amperes.}$$

The vector representing this quantity is drawn to scale in the diagram, making an angle of 9.693 degrees with OE_0. The result-

ant generator current for 100 amperes load current at unity power-factor is given then by the vector sum at OI, and is

$$I = 128.3\underline{/58.8 \text{ degrees}} \text{ amperes.}$$

By a method identical with that followed for the voltage diagram, the generator current for any value of load current at unity power-factor is obtained by taking the proportionate value of the vector AI. If the load power-factor changes, the vector AI is rotated through an angle equal to the power-factor angle, so that solutions for all values of load current between 0 and 100 amperes are determined by the intersections of the circular arcs corresponding to the particular numerical value of current and the line drawn from A making an angle equal to the power-factor angle with AI. The dotted lines are the vectors for a load current of 100 amperes, lagging 25 degrees behind the load voltage.

77. Power-Factor from Vector Diagrams. — The power-factor angle at the generator is equal to the difference between the angles which E and I make with E_0. Thus, where $E = 54{,}300\underline{/38.2 \text{ deg.}}$ and $I = 128.3\underline{/58.8 \text{ deg.}}$ the difference in the angles is 20.6 degrees. The angle of I is the greater, and thus the current is ahead of the voltage in time-phase position. The power-factor at the generator is $\cos 20°.6 = 0.936$. The volt-amperes at the generator $= ei$, and the power in watts $= ei \cos \theta$.

78. Operating Characteristics of Line with Various Loads. — In Table XV are collected a number of numerical values, taken from the vector diagrams by measurement, to illustrate the characteristics of the line under consideration. The values are all based upon a constant load voltage of 50,000 volts to neutral. Three power-factors at the load are considered, 25 deg. lagging, 0 deg. lag, and 25 deg. leading. Since the numerical values are obtained graphically, the last significant figure may be slightly in error. The curves in Figs. 35, 36, 37, 38, 39, and 40 are plotted from the values tabulated in Table XV.

As would be expected, the generator voltage increases most rapidly (to maintain E_0 constant) in the case of the lagging load, and least rapidly for the leading load. In fact, the generator voltage nearly doubles in the former case, while in the latter case the increase is only 33 per cent for a range of I_0 from 0 to 100 amperes.

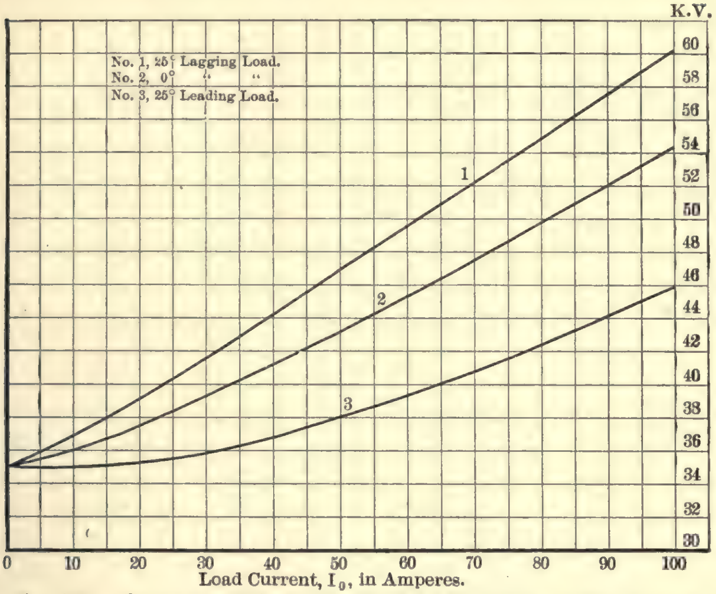

Fig. 35. — Generator Voltage Required to Maintain E_0 Constant at 50,000 Volts.

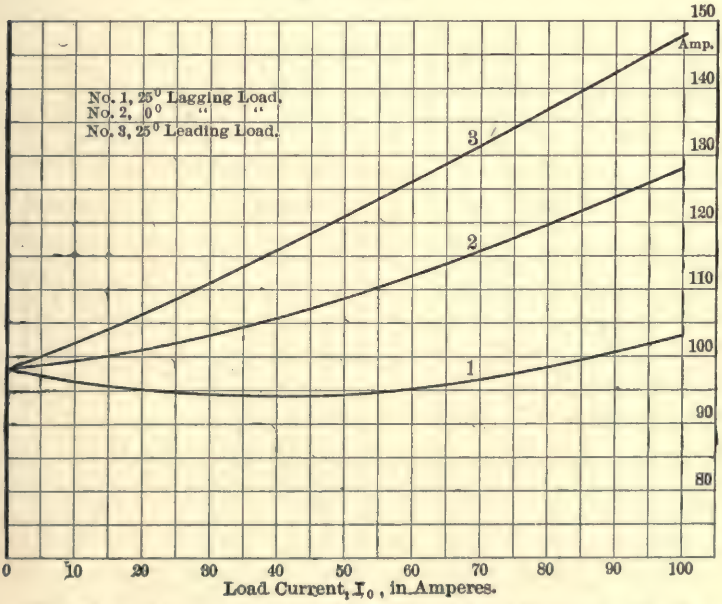

Fig. 36. — Generator Current for Different Load Currents with E_0 Constant at 50,000 Volts.

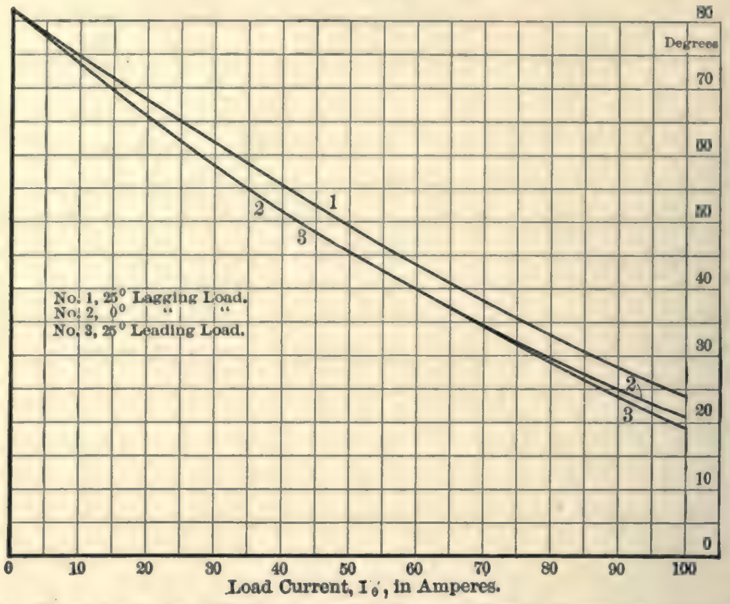

FIG. 37. — Angle by Which Generator Current Leads Generator Voltage. Load Voltage Constant at 50,000 Volts.

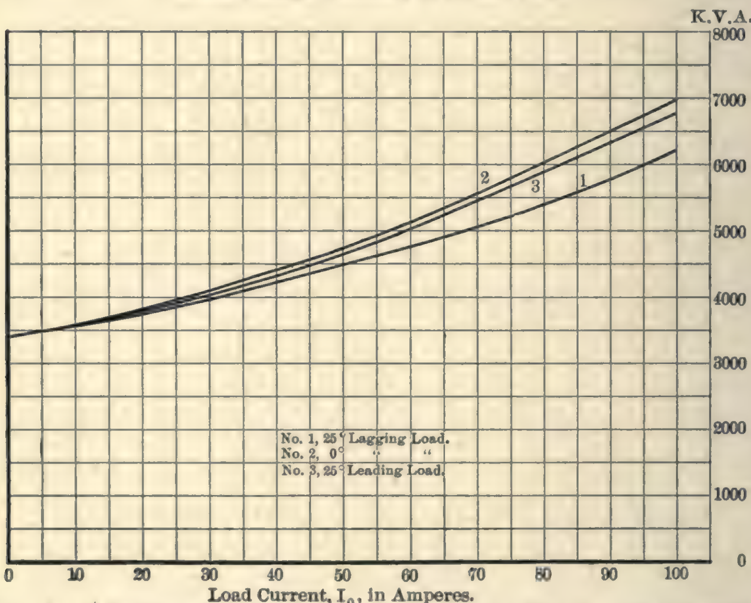

FIG. 38. — Kilovolt-amperes at the Generator with Load Voltage Constant at 50,000 Volts.

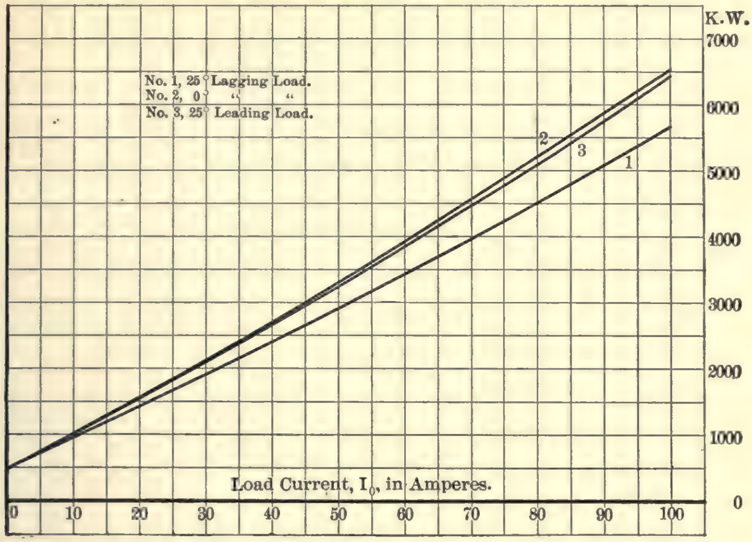

FIG. 39. — Power at Generator for E_0 Constant at 50,000 Volts.

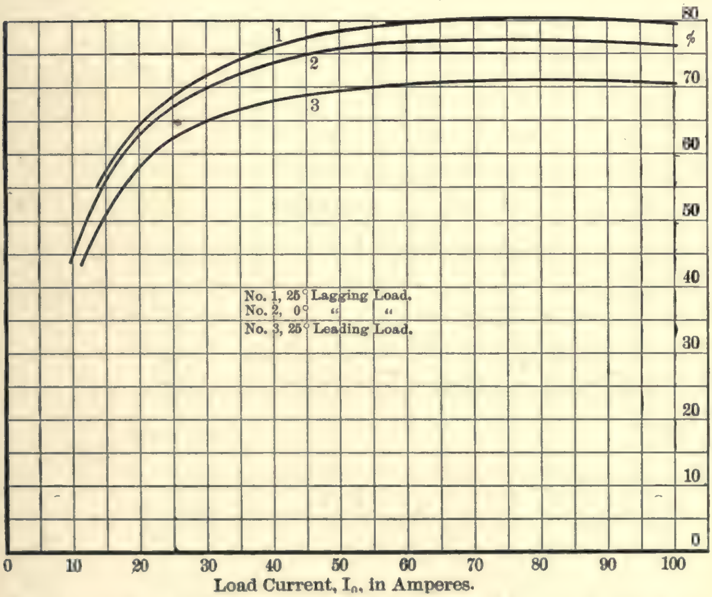

FIG. 40. — Efficiency of Transmission for E_0 Constant at 50,000 Volts.

TABLE XV

OPERATING CHARACTERISTICS OF 400 MILE TRANSMISSION LINE. CONSTANT LOAD VOLTAGE

Numerical Values taken from Vector Diagram

$E_0 = 50{,}000$ volts

I_0, amps.	θ_e, degrees	θ_i, degrees	$\theta = \theta_i - \theta_e$, degrees	P.F. = $\cos\theta$	E, volts	I, amps.	K.V.A. at gen.	K.W. at gen.	K.W. at load	Effic., per cent
1. Load current lagging 25 degrees behind load voltage. P.F. = 0.9063										
0	9.693	91.046	81.353	0.15035	34,510	98.26	3391	509.8	0	0
20	14.9	82.8	67.9	0.376	39,250	95.2	3740	1408	906.3	64.4
40	19.2	74.4	55.2	0.571	44,300	94.2	4170	2380	1812.6	76.2
60	22.4	66.0	43.6	0.724	49,500	95.3	4720	3420	2718.9	79.4
80	25.0	58.2	33.2	0.837	54,800	98.2	5380	4500	3625.0	80.6
100	27.3	50.8	23.5	0.917	60,200	102.9	6200	5690	4532.0	79.6
2. Load current in phase with load voltage. P.F. = 1.0000										
0	9.693	91.046	81.353	0.15035	34,510	98.26	3391	509.8	0	0
20	17.7	83.0	65.3	0.418	37,500	101.1	3800	1590	1000	62.9
40	24.4	76.0	51.6	0.621	41,200	106.0	4370	2715	2000	73.6
60	29.8	69.5	39.7	0.769	45,250	112.3	5080	3910	3000	76.8
80	34.5	63.7	29.2	0.873	49,700	119.7	5950	5200	4000	76.9
100	38.2	58.8	20.6	0.936	54,300	128.3	6960	6520	5000	76.6
3. Load current 25 degrees ahead of load voltage. P.F. = 0.9063										
0	9.693	91.046	81.353	0.15035	34,510	98.26	3391	509.8	0	0
20	19.3	84.7	65.4	0.416	35,200	106.4	3750	1560	906.3	58.1
40	28.2	79.4	51.2	0.627	36,800	115.9	4260	2675	1812.6	67.8
60	36.1	74.8	38.7	0.780	39,200	125.9	4940	3860	2718.9	70.3
80	43.2	71.2	28.0	0.883	42,250	136.9	5780	5100	3625.0	71.0
100	49.2	68.0	18.8	0.947	45,900	148.0	6790	6440	4532.0	70.3

In Fig. 36, showing the generator current, it is seen that at a load power-factor angle of 25 degrees lag, the generator current is practically constant throughout the 100 amperes range of load current. The total range of generator current is only from 94.2 amp. to 102.9 amp. — a maximum variation of 9.2 per cent. At this condition of load, the result of the power transmission is to effect a constant current to a constant voltage transformation. For in-phase and leading load currents, the generator current increases quite uniformly, though by no means as rapidly as the load current. As shown by Fig. 37, within the range of 25 deg. lag to 25 deg. lead, the generator power-factor depends only very slightly upon the load power-factor, but does depend upon the load current.

The transmission efficiencies as shown in Fig. 40 reach a maximum value in all three cases at a load current of from 70 to 80 amperes. As would be expected, the efficiency is greatest for the lagging load — the I^2R losses in the line being greatly reduced on account of the diminished resultant value of line current, the lagging component of the load current neutralizing, in part, the leading capacity current of the line. Of course, if the leakage losses in the line were considerably greater (increased conductance g), the increased E^2g loss, on account of the greatly increased voltage required at the lagging load power-factor, would cause the efficiency in this case to be lower, and in extreme cases, lower, even, than the efficiency at unity and leading power-factors.

79. Constant Generator Voltage; Rigorous Solution. — In the preceding section we have discussed a very convenient method for determining the performances of a transmission system when the load voltage and load current are both known — the convenient graphical processes being based upon the explicit equa- (19). If the problem be stated differently — say constant generator voltage and known load currents — it would appear that no such convenient solution is possible. The knowns in such a case are load current and power-factor, and generator voltage. We are not, in fact, at liberty to specify both the load current and generator voltage throughout unlimited ranges, for we might select a value of load current greater than the generator voltage could supply over a short-circuit at the load, even. The second of equations (19), Chapter 2, when written

$$I_0 = \frac{E}{Z_0 \cosh Vl + U \sinh Vl}, \tag{21}$$

to conform with the notation used in the alternating current problem, can be made to yield a solution, though not without involved expressions and tedious numerical operations. The procedure is as follows: the numerical values of E and I_0 are known as scalars; e being the fixed generator voltage and i_0 the value of the load current for which a solution is desired. The ratio of the two components of the effective load impedance Z_0 is known, for it is this ratio which determines the load power-factor. The numerical value of Z_0 is unknown, and its determination constitutes the essential part of the problem, viz., the determination of the load voltage and the generator current.

Let

$$\phi_0 = \tan^{-1}\frac{x_0}{r_0}.$$

Then the load current lags by the angle ϕ_0 behind the load voltage, and

$$Z_0 = z_0 \cos\phi_0 + jz_0 \sin\phi_0 = z_0\underline{/\phi_0}, \tag{22}$$

where z_0 is the unknown scalar value of Z_0. With z_0 retained as an unknown and the angle ϕ_0 known, the product

$$\left.\begin{aligned} &Z_0 \cosh Vl = z_0\,(a + jb), \\ &\text{in which } a \text{ and } b \text{ are known, and also} \\ &U \sinh Vl = c + jd \end{aligned}\right\} \tag{23}$$

is known, so that

$$Z_0 \cosh Vl + U \sinh Vl = z_0 a + c + j\,(z_0 b + d). \tag{24}$$

By (21), the scalar value of $\dfrac{E}{I_0}$ is equal to the scalar value of the expressions in (24), so that

$$\left(\frac{e}{i_0}\right)^2 = (z_0 a + c)^2 + (z_0 b + d)^2,$$

or

$$z_0^2\,(a^2 + b^2) + z_0\,(2\,ac + 2\,bd) + c^2 + d^2 - \left(\frac{e}{i_0}\right)^2 = 0,$$

which gives

$$z_0 = \frac{-\,(ac + bd) \pm \sqrt{(ac + bd)^2 - (a^2 + b^2)\left[c^2 + d^2 - \left(\dfrac{e}{i_0}\right)^2\right]}}{a^2 + b^2}. \tag{25}$$

The positive value of z_0 from (25) is to be retained. Failure to secure such a positive value indicates that a value of i_0 has been used which is greater than the generator voltage e could supply. Having the numerical value of z_0, the load voltage is immediately obtained by

$$E_0 = z_0 (\cos \phi_0 + j \sin \phi_0) I_0,$$

where I_0 may be used as reference vector. Then E and I are obtained by equations (19) in their proper complex quantity notation by using I_0 and E_0 as found. A good check on the accuracy of the numerical work is afforded by the fact that the recomputed scalar value of generator voltage must agree with the value assumed at the outset. Of course, it is not necessary to recompute the value of E unless it is desired to check the work or to secure values of generator power, in which case it is necessary to do so in order to secure the power-factor angle at the generator.

80. Constant Generator Voltage; Solution by Use of Plotted Curves. — The procedure above outlined is not at all adapted to the computation of data for curve plotting, but it, or its equivalent, must be followed if results more accurate than could be read from curves are desired. If plotted curves are sufficiently accurate, then the solutions for constant generator voltage may be obtained with great ease by an indirect process, based upon the results obtained for constant load voltage in the preceding paragraphs.

By equation (21) we see that I_0 varies directly as E as long as the equivalent load impedance Z_0 remains constant. $E_0 = Z_0 I_0$ thus varies directly as E also. In Table XV, each solution as tabulated for a given value of I_0 corresponds to some particular value of Z_0, so that if we assume that the value of Z_0 remains constant while the generator voltage changes from the tabulated value, obtained by the graphical solution described, to the constant value we wish to consider, the corresponding values of I_0 and E_0 for this latter case can be found by taking them in the same ratio to the original values as the ratio of the new E to the original E. The generator current will also change in the same ratio, while the values of power and volt-amperes will change by the square of this ratio. For example: in the solution for $I_0 = 80$ amperes at a leading power-factor angle of 25 degrees and $E_0 = 50{,}000$ volts,

$E = 42{,}250\underline{/43.2\text{ deg.}}$ and $I = 136.9\underline{/71.2\text{ deg.}}$ If E be changed to 50,000 volts, then by proportionality

$$I = \frac{50{,}000}{42{,}250} \times 136.9 = 162 \text{ amperes.}$$

$$I_0 = \frac{50{,}000}{42{,}250} \times 80.0 = 94.7 \text{ amperes.}$$

$$E_0 = \frac{50{,}000}{42{,}250} \times 50{,}000 = 59{,}200 \text{ volts.}$$

$$P_0 = \left(\frac{50{,}000}{42{,}250}\right)^2 \times 3625 = 5080 \text{ K.W.}$$

$$P = \left(\frac{50{,}000}{42{,}250}\right)^2 \times 5100 = 7140 \text{ K.W.}$$

$$\text{K.V.A.} = \left(\frac{50{,}000}{42{,}250}\right)^2 \times 5780 = 8090 \text{ K.V.A. at gen.}$$

The efficiency, of course, does not change, since both load and generator powers change in the same ratio. It may be remarked here, as will be discussed later, that the efficiency of transmission of a given line depends only upon the equivalent load-end impedance Z_0, and not in any way upon the magnitude of voltage applied at the generator end of the line.

Thus, in general, let

$$K = \frac{\text{new generator voltage}}{\text{former generator voltage}}. \tag{26}$$

Then

$$\left.\begin{aligned} &\text{New currents or voltages} = K\text{ (former currents or voltages),} \\ &\text{New powers or volt-amperes} = K^2\text{ (former powers or volt-} \\ &\qquad\qquad\qquad\qquad\text{amperes).} \end{aligned}\right\} \tag{27}$$

In the transmission line under discussion let the generator voltage be so adjusted that the load voltage is 50,000 volts at a load current of 80 amperes, lagging 25 deg. Its value from Table XV = E = 54,800 volts. Table XVI shows the method of obtaining the data for performance curves under the condition of constant generator voltage from the data for constant load voltage in Table XV.

The curves in Figs. 41 to 46 inclusive have been plotted with load current as abscissæ, or independent variable, from the data in Table XVI.

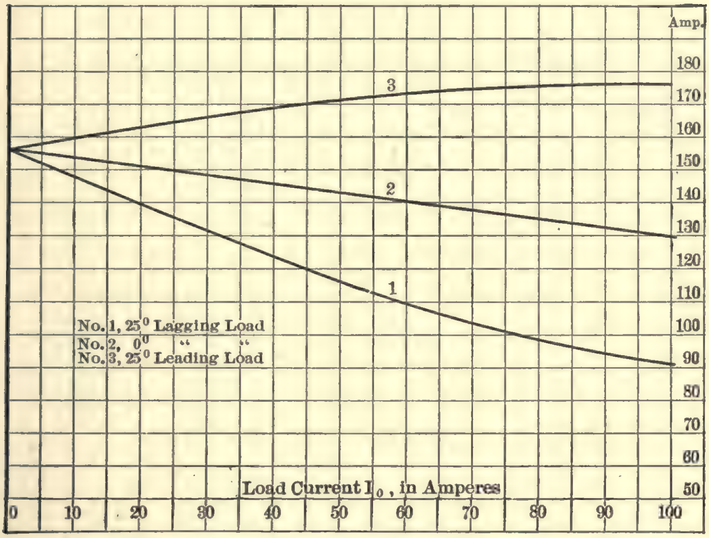

FIG. 41. — Generator Current for Generator Voltage Constant at 54,800 Volts.

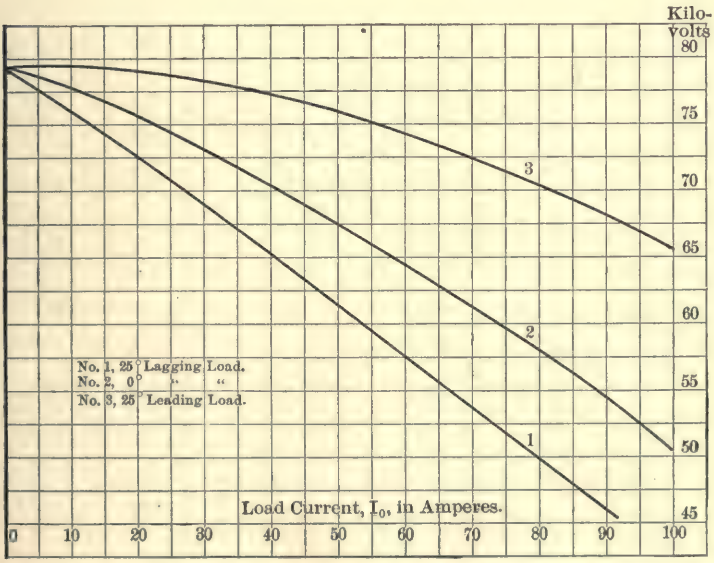

FIG. 42. — Load Voltage with Generator Voltage Constant at 54,800 Volts.

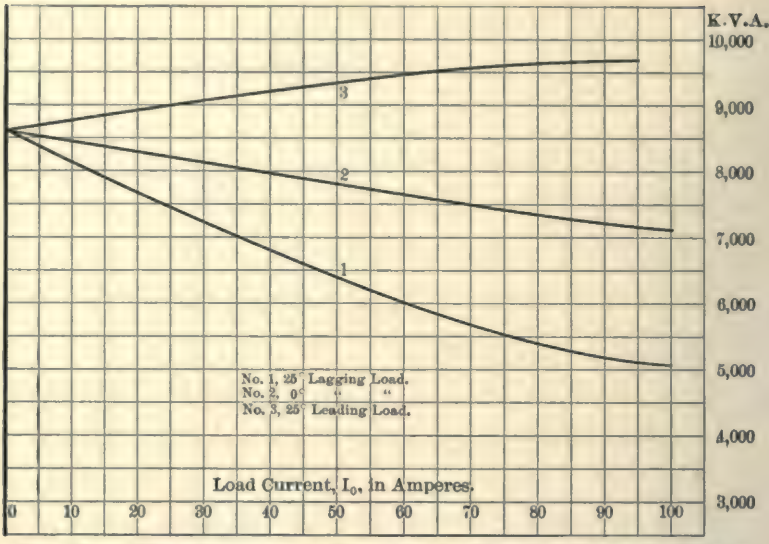

FIG. 43. — Kilovolt-amperes at Generator for Constant Generator Voltage of 54,800 Volts.

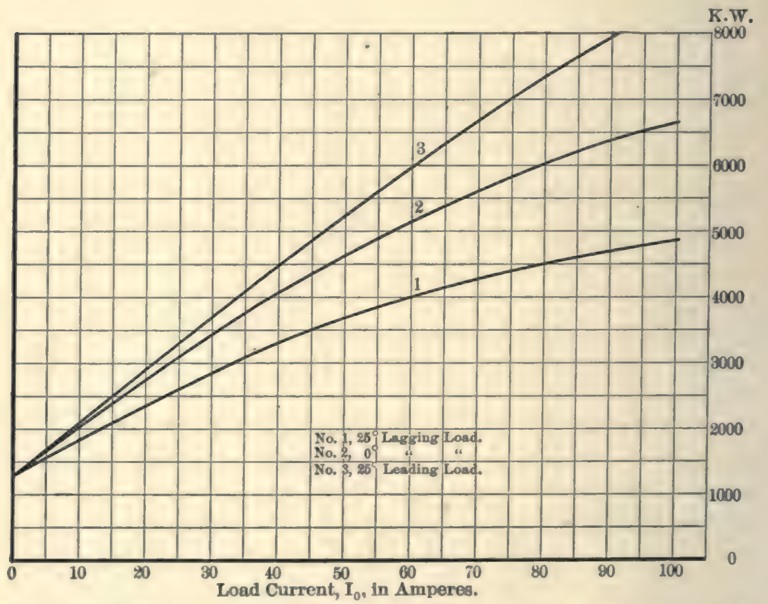

FIG. 44. — Power at Generator for Constant Generator Voltage of 54,800 Volts.

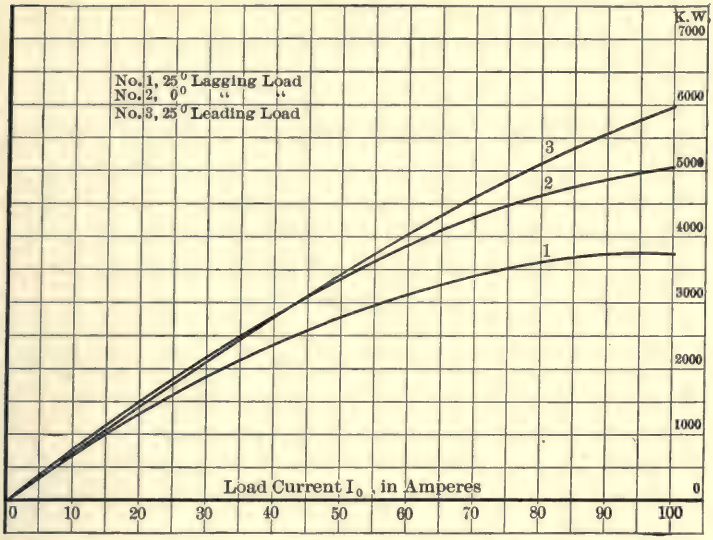

FIG. 45. — Power at Load End with Generator Voltage Constant at 54,800 Volts.

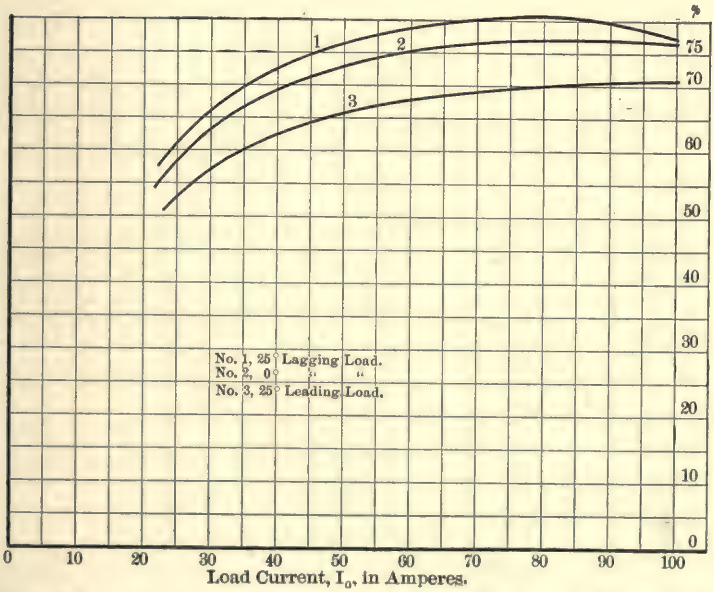

FIG. 46. — Transmission Efficiency with Generator Voltage Constant at 54,800 Volts.

TABLE XVI

Operating Characteristics of 400 Mile Transmission Line at Constant Generator Voltage

Generator voltage = 54,800 volts = E. Computed on basis of values given in Table XV by equations (26) and (27).
E' = gen. voltage from Table XV for E_0 = 50,000 volts

E', volts	$K = \frac{E}{E'}$	I, amps.	I_0, amps.	E_0, volts	K.V.A. at gen.	K.W. at gen.	K.W. at load	Effic., per cent
1. Load current lagging 25 degrees behind load voltage. P.F. = 0.9063								
34,510	1.587	156.0	0	79,400	8570	1283	0	0
39,250	1.394	132.9	27.9	69,800	7280	2750	1763	64.4
44,300	1.236	116.6	49.5	61,900	6390	3650	2780	76.2
49,500	1.106	105.3	66.4	55,300	5790	4190	3320	79.4
54,800	1.000	98.2	80.0	50,000	5380	4500	3625	80.6
60,200	0.910	93.6	91.0	45,500	5130	4710	3750	79.6
2. Load current in phase with load voltage. P.F. = 1.0000								
34,510	1.587	156.0	0	79,400	8570	1283	0	0
37,500	1.460	147.9	29.2	73,000	8120	3400	2130	62.9
41,200	1.330	141.0	53.2	66,500	7740	4810	3550	73.6
45,250	1.210	136.0	72.7	60,500	7460	5730	4410	76.8
49,700	1.101	131.9	88.2	55,100	7230	6330	4870	76.9
54,300	1.010	129.6	101.0	50,500	7100	6650	5100	76.6
3. Load current 25 degrees ahead of load voltage. P.F. = 0.9063								
34,510	1.587	156.0	0	79,400	8570	1283	0	0
35,200	1.555	165.6	31.1	77,900	9090	3780	2200	58.1
36,800	1.489	172.5	59.5	74,500	9470	5940	4020	67.8
39,200	1.398	176.0	83.9	69,900	9670	7560	5320	70.3
42,250	1.296	177.3	103.7	64,900	9710	8600	6100	71.0
45,900	1.193	176.8	119.3	59,700	9690	9200	6470	70.3

81. Infinite Series Form for Transmission Line Solutions. — For short lines operating at low frequencies and having only moderate power losses, the solutions as given in equations (3) and (19) may be expressed by infinite series, the successive terms in which decrease rapidly. Equation (19) is perhaps more easily expanded than equation (3), so the former will be used. The series for the hyperbolic cosine and sine are

$$\left.\begin{aligned}\cosh x &= 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \frac{x^6}{\underline{|6}} + \cdots,\\ \sinh x &= x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \frac{x^7}{\underline{|7}} + \cdots.\end{aligned}\right\} \qquad (28)$$

Introducing these series in (19) with

$$x = Vl = \sqrt{ZY}l = \sqrt{Z_tY_t},$$

where

$$\left.\begin{aligned}Z_t &= Zl,\\ Y_t &= Yl,\\ \sqrt{\frac{Z_t}{Y_t}} &= \sqrt{\frac{Z}{Y}},\end{aligned}\right\} \qquad (29)$$

and

we have immediately

$$E = E_0\left(1 + \frac{Z_tY_t}{2} + \frac{Z_t^2Y_t^2}{24} + \cdots\right) + I_0\sqrt{\frac{Z_t}{Y_t}}\left(\sqrt{Z_tY_t} + \frac{(Z_tY_t)^{\frac{3}{2}}}{6} + \cdots\right),$$

$$I = I_0\left(1 + \frac{Z_tY_t}{2} + \frac{Z_t^2Y_t^2}{24} + \cdots\right) + E_0\sqrt{\frac{Y_t}{Z_t}}\left(\sqrt{Z_tY_t} + \frac{(Z_tY_t)^{\frac{3}{2}}}{6} + \cdots\right),$$

which, by combining factors in the second portion of each, become

$$\left.\begin{aligned}E &= E_0\left(1 + \frac{Z_tY_t}{2} + \frac{Z_t^2Y_t^2}{24} + \frac{Z_t^3Y_t^3}{720} + \cdots\right)\\ &\qquad + I_0Z_t\left(1 + \frac{Z_tY_t}{6} + \frac{Z_t^2Y_t^2}{120} + \frac{Z_t^3Y_t^3}{5040} + \cdots\right),\\ I &= I_0\left(1 + \frac{Z_tY_t}{2} + \frac{Z_t^2Y_t^2}{24} + \frac{Z_t^3Y_t^3}{720} + \cdots\right)\\ &\qquad + E_0Y_t\left(1 + \frac{Z_tY_t}{6} + \frac{Z_t^2Y_t^2}{120} + \frac{Z_t^3Y_t^3}{5040} + \cdots\right).\end{aligned}\right\} \qquad (30)$$

Of course, Z_t and Y_t must be used in their complex form. The rapidity of convergence of these series is dependent upon the value of the argument Z_tY_t, and thus upon the square of the product

of the line length and frequency, and also, to a lesser extent, upon the product of total line conductance and total line resistance.

The form of any type of solution is worthy of attention. The generator voltage, or current, is always expressed in terms of the load voltage and current by linear equations of the form

$$\left.\begin{aligned} E &= AE_0 + BI_0, \\ I &= AI_0 + CE_0, \end{aligned}\right\} \tag{31}$$

in which the constants A, B, and C are functions of the physical properties of the system and the frequency only, and not of the voltages or currents. In effect, all the various methods for transmission line solutions are only schemes for determining the values of these three constants. Having them, the solutions for various loads are obtained with as much ease in the case of a long line as in the case of a short one. In equation (31) the functions

$$\cosh Vl, \quad U \sinh Vl, \quad \text{and} \quad \frac{1}{U} \sinh Vl$$

are the values of A, B, and C, respectively. In equation (30), the series are again merely expressions for these constants. It may be remarked here that the current and voltage at any point in a complicated network of circuits containing but one source of energy can always be expressed as such linear functions of the voltage and current at any other point, as long as the physical properties of the different parts of the system are pure constants and independent of the voltage or current. By far the most advantageous procedure in the calculation of the performances of any such system is to calculate first the values of these constants and then to apply the resulting numerical expressions to the particular numerical values of E_0 and I_0 for which a solution is desired. Further, expression in this form always leads to the convenient graphical process of calculation as just described in application to the long transmission line.

82. Illustration of Convergence of Series. — To illustrate the manner of convergence of the above series expressions for A, B, and C, the following table shows their values, for the 400 mile line under consideration, when different numbers of terms are included in the parentheses.

TABLE XVII

CONVERGENCE OF SERIES EXPRESSION FOR THE CONSTANTS A, B, AND C IN SOLUTION FOR 400 MILE TRANSMISSION LINE

Number of terms used	A	B	C
1	$1.00000\underline{/0°.000}$	$326.70\underline{/70°.324}$	$0.0022024\underline{/88°.439}$
2	$0.67732\underline{/11.093}$	$290.53\underline{/73.124}$	$0.0019586\underline{/91.239}$
3	$0.69035\underline{/\ 9.652}$	$291.52\underline{/72.927}$	$0.0019653\underline{/91.042}$
4	$0.69020\underline{/\ 9.694}$	$291.51\underline{/72.931}$	$0.0019652\underline{/91.046}$
Inf.	$0.69020\underline{/\ 9.693}$	$291.51\underline{/72.932}$	$0.0019652\underline{/91.046}$

An inspection of the tabulated values shows that, as far as engineering calculations based upon the more or less uncertain values of the line constants are concerned, the use of three terms in the series expression yields results, in this case, which are sufficiently close to the true values as given by the use of the hyperbolic functions (infinite number of terms). In the case of the B and C coefficients, two terms give very accurate results, but on account of the slower convergence of the hyperbolic cosine series, it is necessary to use at least three terms in the calculation of the coefficient A.

In spite of the rapid convergence of these series expressions, the amount of labor involved in their use is much greater than for a rigorous solution based upon the hyperbolic functions directly, as in equation (19), provided tables of hyperbolic functions of real variables are available. Such being the case, there is no real need of setting up criteria for estimating, before the numerical computation is undertaken, the number of terms which must be used in the series to give a desired degree of approximation. The writer is well aware of the aversion of the general engineering public to the use of such hyperbolic function forms of expressions as equations (19), but, when such expressions are so easily dealt with numerically and at the same time furnish results concerning which there can be no doubt as to the accuracy, it appears that such prejudice is unwarranted.

83. Line Admittance and Impedance Localized. — From purely physical considerations, it appears that the effect of the distributed admittance of a transmission line can be approximated by placing a lumped admittance across the line at the mid-point

of its length, equal in value to the total amount of the distributed admittance. Doing this, the equivalent circuit representing the transmission line is as in Fig. 47.

In this circuit,

$$E' = \text{voltage at mid-point} = E_0 + \tfrac{1}{2} Z_t I_0,$$
$$I' = \text{current through admittance} = E'Y_t = Y_t E_0 + \tfrac{1}{2} Z_t Y_t I_0.$$
$$\left.\begin{aligned} I &= I_0 + I' = I_0 (1 + \tfrac{1}{2} Z_t Y_t) + E_0 Y_t, \\ E &= E' + \tfrac{1}{2} Z_t I = E_0 (1 + \tfrac{1}{2} Z_t Y_t) + I_0 Z_t (1 + \tfrac{1}{4} Z_t Y_t). \end{aligned}\right\} \quad (32)$$

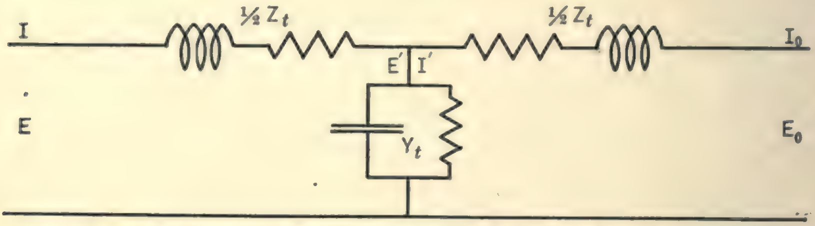

FIG. 47. — Approximate Representation of Distributed Line Admittance.

In equation (32), based on the approximation that the admittance of the line is all concentrated at the middle, the constants in the general form of equation (31) have the values,

$$\left.\begin{aligned} A &= (1 + \tfrac{1}{2} Z_t Y_t), \\ B &= Z_t (1 + \tfrac{1}{4} Z_t Y_t), \\ C &= Y_t. \end{aligned}\right\} \quad (33)$$

Comparing these values with the series, equation (30), we see that the expression for A consists of the first two terms in the series for the hyperbolic cosine. The expression for B is also the first two terms in the series for $U \sinh Vl$, except that the coefficient of $Z_t Y_t$ in the parentheses above is one-fourth, instead of one-sixth, as in the true series. The expression for C in (33) is equivalent to the series expression in (30) when the first term only (unity) is retained in the parentheses. The amount of computation involved in using equations (33) is the same as in using the first two terms only of the rigorous series expressions (30), and the results are not as accurate, because the values of the coefficients in the B and C terms are not the same as in the series. Therefore there is no possible excuse for using the approximate expressions based upon a concentration of the total admittance at the middle

of the line, except, perhaps, in an academic way, to impress upon students the physical significance of leakage and capacity effects.

Another approximation method is based upon a different concentration of the line admittance. It assumes that the distributed line admittance may be replaced by locating two-thirds of the total admittance at the mid-point of the line, and one-sixth of the total at each end. This method is equivalent to assuming that the electrical quantities are distributed along the line in a way represented by an arc of a parabola. (See Steinmetz' "Alternating

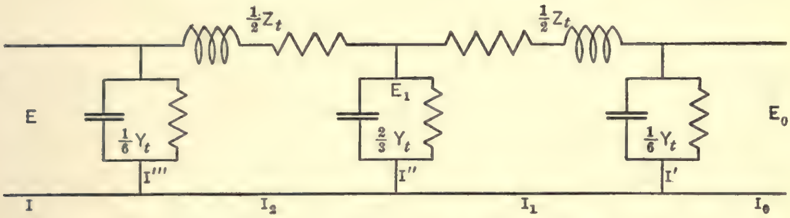

Fig. 48. — Approximate Representation of Distributed Line Admittance.

Current Phenomena," page 225.) Figure 48 shows the arrangement. Using the notation shown in the diagram,

$$I_1 = I_0 + I' = I_0 + E_0 \tfrac{1}{6}\, Y_t.$$

$$E_1 = E_0 + I_1 \tfrac{1}{2}\, Z_t = E_0 \left(1 + \tfrac{1}{12}\, Z_t Y_t\right) + \tfrac{1}{2}\, Z_t I_0.$$

$$I'' = \tfrac{2}{3}\, Y_t E_1 = E_0 Y_t \left(\tfrac{2}{3} + \tfrac{2}{36}\, Z_t Y_t\right) + \tfrac{1}{3}\, Z_t Y_t I_0.$$

$$I_2 = I_1 + I'' = E_0 Y_t \left(\tfrac{5}{6} + \tfrac{2}{36}\, Z_t Y_t\right) + I_0 \left(1 + \tfrac{1}{3}\, Z_t Y_t\right).$$

$$E = E_1 + \tfrac{1}{2}\, Z_t I_2. \qquad I = I_2 + \tfrac{1}{6}\, Y_t E.$$

$$\left.\begin{aligned} E &= E_0\left(1 + \frac{Z_tY_t}{2} + \frac{Z_t^2Y_t^2}{36}\right) + I_0 Z_t\left(1 + \frac{Z_tY_t}{6}\right), \\ I &= I_0\left(1 + \frac{Z_tY_t}{2} + \frac{Z_t^2Y_t^2}{36}\right) + E_0 Y_t\left(1 + \frac{5}{36} Z_tY_t + \frac{Z_t^2Y_t^2}{216}\right). \end{aligned}\right\} \qquad (34)$$

Naturally, equations (34) are closer approximations to the series in equations (30) than are those developed on the assumption that the line admittance may all be concentrated at the middle. In the above equation, the expression for the coefficient A differs from that in the series by only

$$\frac{Z_t^2Y_t^2}{72} + \text{higher power terms,}$$

and is therefore quite accurate. The expression for the coefficient B does not contain the term in $Z_t^2Y_t^2$, nor any above this order, but the coefficient of the Z_tY_t term is the same as in the series. In the expression for the constant C the coefficients of both the Z_tY_t and the $Z_t^2Y_t^2$ terms are smaller than in the series. The labor involved in computing from equations (34) is the same as from the series expressions, when the same number of terms are included, so there is no reason for using the former, since they are not as accurate as the series when the latter are written to the same number of terms.

To summarize, it is not possible to find any method of obtaining solutions for the type of alternating current transmission lines under consideration which surpasses, in ease of application, conciseness of expression, or in accuracy of results, the general and rigorous formulæ expressed in terms of hyperbolic functions. Even the simplest approximation — single equivalent admittance at the middle of the line — does not materially lessen the work, and, as pointed out, is quite inaccurate — absolutely so when the line approaches a quarter-wave length.

84. Collection of Formulæ for Use in Computing. Tabular Arrangement. — For the convenience of computers, the formulæ and tabular form for the computation leading to the determination of the constants A, B, and C, in the general form of solution (31) through the use of the rigorous hyperbolic function expressions (19), are collected here.

Collection of Formulæ

r = line resistance per unit length, ohms.

g = line leakage per unit length, mhos.

L = line self-inductance per unit length, henrys.

C = line electrostatic capacity per unit length, farads.

f = frequency in cycles per second.

l = line length in units used above.

$x = 2\pi fL. \quad Z = r + jx = z/\underline{\theta_z}.$

$\theta_z = \tan^{-1}\dfrac{x}{r}. \quad z = \dfrac{r}{\cos\theta_z} = \dfrac{x}{\sin\theta_z} = \sqrt{r^2 + x^2}.$

$b = 2\pi fC. \quad Y = g + jb = y/\underline{\theta_y}.$

$\theta_y = \tan^{-1}\dfrac{b}{g}. \quad y = \dfrac{g}{\cos\theta_y} = \dfrac{b}{\sin\theta_y} = \sqrt{g^2 + b^2}.$

$V = \sqrt{ZY} = v/\underline{\theta_v} = \alpha + j\beta. \quad v = \sqrt{zy}. \quad 2\theta_v = \theta_z + \theta_y.$

$\alpha = v\cos\theta_v. \quad \beta = v\sin\theta_v.$

$U = \sqrt{\dfrac{Z}{Y}} = u/\underline{\theta_u}. \quad u = \sqrt{\dfrac{z}{y}}. \quad 2\theta_u = \theta_z - \theta_y.$

$A = \cosh Vl = \cosh\alpha l\cos\beta l + j\sinh\alpha l\sin\beta l = a/\underline{\theta_a}.$

$S = \sinh Vl = \sinh\alpha l\cos\beta l + j\cosh\alpha l\sin\beta l = s/\underline{\theta_s}.$

(Express $\cosh Vl$ and $\sinh Vl$ in the form of scalars with their angles in the same way as was done for the impedance Z and admittance Y above).

$B = U\sinh Vl = b/\underline{\theta_b}. \quad b = us. \quad \theta_b = \theta_u + \theta_s.$

$C = \dfrac{1}{U}\sinh Vl = c/\underline{\theta_c}. \quad c = \dfrac{s}{u}. \quad \theta_c = \theta_s - \theta_u.$

The values of the three constants A, B, and C, to be used in the general form of equation (31), are obtained by the above formulæ. Of course, the symbol C as used in these equations is not to be confused with the same symbol when used to indicate electrostatic capacity.

TABLE XVIII

TABULAR FORM FOR COMPUTING FROM FORMULÆ ON PRECEDING PAGE

r	6.560
g	0.100×10^{-6}
L	0.003788
C	0.00790×10^{-6}
f	800 cycles
l	200 miles
$\log 2\pi$	0.79818
$\log f$	2.90309
$\log L$	7.57841–10
$\log 2\pi f$	3.70127
$\log C$	1.89763–10
$\log x$	1.27968
$\log r$	0.81690
$\log \tan \theta_z$	0.46278
$\log \sin \theta_z$	9.97564–10
$\log z$	1.30404
θ_z	70.990 deg.
$\log b$	5.59890–10
$\log g$	3.00000–10
$\log \tan \theta_y$	2.59890
$\log \sin \theta_y$	0.00000
$\log y$	5.59890–10
θ_y	89.856 deg.
$\log v^2$	6.90294–10
$2\,\theta_v$	160.846 deg.
θ_v	80.423 deg.
$\log \cos \theta_v$	9.22108–10
$\log v$	8.45147–10
$\log \sin \theta_v$	9.99390–10
$\log \alpha$	7.67255–10
$\log \beta$	8.44537–10
$\log 360/2\pi$	1.75812
$\log \beta^\circ$	0.20349
α	0.0047049
β°	1.59768 deg.
αl	0.94098
$\beta^\circ l$	319.536 deg.
$\log z/y$	5.70514
$2\,\theta_u$	−18.866 deg.
$U \begin{cases} \log u \end{cases}$	2.85257
θ_u	− 9.433 deg.

$\log \cosh \alpha l$	0.16919
$\log \sinh \alpha l$	0.03588
$\log \cos \beta l$	+9.88128
$\log \sin \beta l$	−9.81223–10
$\log \cosh \alpha l \cos \beta l$	+0.05047
$\log \sinh \alpha l \sin \beta l$	−9.84811–10
$\log \tan \theta_a$	−9.79764–10
$\log \cos \theta_a$	9.92790–10
$A = \cosh Vl$ $\log a$	0.12257
θ_a	327.890 deg.
$\log \sinh \alpha l \cos \beta l$	+9.91716–10
$\log \cosh \alpha l \sin \beta l$	−9.98142–10
$\log \tan \theta_s$	−0.06426
$\log \sin \theta_s$	−9.87924–10
$\sinh Vl$ $\log s$	0.10218
θ_s	310.777 deg.
$B = U \sinh Vl$ $\log b$	2.95475
θ_b	301.344 deg.
$C = \frac{1}{U} \sinh Vl$ $\log c$	7.24961–10
θ_c	320.210 deg.

NUMERICAL RESULTS. Equation 31

A	1.3261	/327.890 deg.
B	901.1	/301.344 deg.
C	0.001777	/320.210 deg.

A minus sign before the logarithm means that the quantity represented by the logarithm is negative.

The numerical values entered in the table correspond to a line whose constants are those given in the first six entries of the table, and are approximately those for No. 8 B. & S. gauge wire spaced 18 inches. See "Standard Handbook." Constants for loop of two wires.

CHAPTER VII

THE PROPAGATION CONSTANT, $V = \sqrt{ZY} = \alpha + j\beta$

85. General. — In the preceding sections it has been shown how to obtain the numerical values of the quantity $V = \alpha + j\beta$ which enters into the exact solution for a line carrying an alternating current. In Chapter VI it is seen that α, the real portion of V, is the term which accounts for the decrease in the magnitude of the component waves in the direction of their propagation. For this reason the name "Attenuation Constant" has been given to the quantity α. The quantity β, as the imaginary portion of V, determines the amount of rotation of the vectors representing the component waves for each unit of line length. β is numerically the rotation of each vector, in radians, per unit length. Since a complete rotation of one of these component vectors takes place in a distance along the line equal to one wave length, the constant β, which determines this distance, is termed the "Wave-length Constant." The combination of the two quantities into the single complex, $V = \alpha + j\beta$, is known as the "Propagation Constant," since this quantity completely determines the manner of propagation along the line of the separate component waves, and thus their resultant. Both components of V are dependent upon all of the line properties — resistance, self-inductance, leakage, electrostatic capacity, and frequency.

86. Fundamental Equations for α and β. — The attenuation constant, α, depends, in the main, upon the line resistance and leakage, but its value is also affected by the other three properties.

By equation (5), Chapter VI,

$$\alpha = \sqrt{\tfrac{1}{2}\,(zy - xb + rg)}, \tag{1}$$

which, when expanded by

$$x = \omega L, \qquad b = \omega C, \qquad z = \sqrt{r^2 + x^2}, \qquad y = \sqrt{g^2 + b^2},$$

becomes

$$\alpha = \sqrt{\tfrac{1}{2}\left\{\sqrt{(r^2 + \omega^2 L^2)(g^2 + \omega^2 C^2)} + rg - \omega^2 LC\right\}}. \tag{2}$$

The quantity β depends principally upon the frequency, self-inductance, and capacity, though it is not independent of resistance and leakage. By equation (5), Chapter VI,

$$\beta = \sqrt{\tfrac{1}{2}\,(zy + xb - rg)}, \tag{3}$$

which by the above expansion becomes

$$\beta = \sqrt{\tfrac{1}{2}\left\{\sqrt{(r^2 + \omega^2 L^2)\,(g^2 + \omega^2 C^2)} - rg + \omega^2 LC\right\}}. \tag{4}$$

The manner of variation of these two quantities, α and β, with variation of the several separate quantities entering into their determination, has been extensively studied by Pupin, Fleming, Heaviside, and others, on account of their importance in determining the operating characteristics of long electric cables.

If $f = 0$, $\omega = 0$, and the expressions reduce to

$$\alpha\Big]_{\omega=0} = \sqrt{rg}. \qquad \beta\Big]_{\omega=0} = 0, \tag{5}$$

Thus, $V\Big]_{\omega=0} = \sqrt{rg} + j\,0 = v$ as used in the discussion of direct current transmission over leaky lines, Chapter II. If $r = 0$ and $g = 0$, we have immediately

$$\alpha\Big]_{\substack{r=0\\g=0}} = 0; \qquad \beta\Big]_{\substack{r=0\\g=0}} = \sqrt{xb} = 2\,\pi f\sqrt{LC}. \tag{6}$$

87. Wave Length and Velocity of Propagation. — In any case the distance along the line required for a complete rotation of the component vectors is

$$\lambda = \frac{2\,\pi}{\beta} = \text{wave length}, \tag{7}$$

and the velocity of the separate waves is

$$S = f\lambda = \frac{2\,\pi f}{\beta} = \text{velocity of propagation}. \tag{8}$$

Thus, with no line losses ($\alpha = 0$),

$$\left.\begin{aligned} \lambda\Big]_{\alpha=0} &= \frac{2\,\pi}{2\,\pi f\sqrt{LC}} = \frac{1}{f\sqrt{LC}}, \\ \text{and} \qquad & \\ S_\infty = S\Big]_{\alpha=0} &= \frac{2\,\pi f}{2\pi f\sqrt{LC}} = \frac{1}{\sqrt{LC}}. \end{aligned}\right\} \tag{9}$$

The velocity of propagation when there are no line losses is $1/\sqrt{LC}$. This is the limiting value of the velocity, and is equal to the velocity of light $= 3 \times 10^{10}$ cm. per second, for conductors in air, when the internal self-inductance of the metallic conductors is neglected. Except under one special condition, the introduction of energy losses due to resistance of the conductors or leakage between the conductors, will cause the value of S to be smaller than S_∞, although, for all values of r and g the velocity S at continuously increasing frequencies approaches S_∞ as a limit. This approach to S_∞ at infinite frequencies arises from the preponderance of the zy and xb terms in equation (3), so that

$$S\Big]_{\omega=\infty} = \frac{1}{\sqrt{LC}} = S_\infty. \tag{10}$$

Also, at high frequencies, the internal self-inductance of the conductors diminishes, and finally becomes zero, so that at the limit, the velocity, $S\Big]_{\omega=\infty}$, becomes equal to that of light because of L becoming equal to the external self-inductance only. Of course, at these very high frequencies, the radiation of energy into space would introduce losses and thereby increase the effective values of r and g. Since equation (10) was set up under the assumed condition that r and g remain constant, it is, of course, not true in general when r and g are both functions of the frequency. If the product rg increases less than in proportion to the square of the frequency and L and C remain constant, then equation (10) remains true. The subject of radiation losses cannot be taken up here.

88. Values of α, β, and S, when $gL = rC$. If equations (2) and (4) be expanded,

$$\left.\begin{aligned} \alpha &= \left[\tfrac{1}{2}\left\{(r^2g^2 + \omega^2r^2C^2 + \omega^2g^2L^2 + \omega^4L^2C^2)^{\frac{1}{2}} + rg - \omega^2LC\right\}\right]^{\frac{1}{2}}, \\ \beta &= \left[\tfrac{1}{2}\left\{(r^2g^2 + \omega^2r^2C^2 + \omega^2g^2L^2 + \omega^4L^2C^2)^{\frac{1}{2}} - rg + \omega^2LC\right\}\right]^{\frac{1}{2}}, \end{aligned}\right\} \tag{11}$$

which, by adding and subtracting $2\,\omega^2rgLC$ to the term in parentheses (see Fleming, "Propagation of Electric Currents," page 69), reduce to the following form:

$$\left.\begin{aligned} \alpha &= \sqrt{\tfrac{1}{2}\left\{\sqrt{(gr + \omega^2LC)^2 + \omega^2\,(gL - rC)^2} + (gr - \omega^2LC)\right\}}, \\ \beta &= \sqrt{\tfrac{1}{2}\left\{\sqrt{(gr + \omega^2LC)^2 + \omega^2\,(gL - rC)^2} - (gr - \omega^2LC)\right\}}. \end{aligned}\right\} \tag{12}$$

The form in (12) permits of a more ready determination of the manner of variation of the values of α and β than those previously given, but as far as computing purposes are concerned it is open to the same objections as equation (5). If we assume that the fundamental constants r, L, g, and C do not change with the frequency (an assumption sometimes far from the truth), the manner of variation of α and β with the frequency can easily be determined by inspection of equations (12).

If the relation $gL - rC = 0$ be fulfilled, the equations for α and β reduce to

$$\alpha\Big]_{gL=rC} = \sqrt{rg}, \qquad \beta\Big]_{gL=rC} = \omega\sqrt{LC}, \tag{13}$$

indicating that α is independent of the frequency and β is proportional to the frequency. α in this case is equal to the direct-current attenuation constant, and the velocity of propagation

$$S\Big]_{gL=rC} = \frac{1}{\sqrt{LC}} = S_\infty \tag{14}$$

is constant, irrespective of the frequency.

If the relation $gL - rC = 0$ be not fulfilled, the second term in parentheses under the double radical in the expression for α has a value, either positive or negative, and since its square is always positive, the value of α will continuously increase with increasing frequency — from its direct-current value, $\sqrt{rg}$, towards some limiting value at infinite frequency. β^2 increases continuously with increasing frequency, from 0 at zero frequency towards $(\omega^2LC + \text{constant})$ at infinite frequency.

89. Infinite-frequency Values of α, β, and S. — The best way to establish the infinite-frequency values of α and β is to develop their equivalent expressions into power series in $\frac{1}{\omega}$, and then to note the finite terms when $\omega = \infty$. Beginning with the fundamental equations (1) and (3), and letting

$$\frac{r}{x} = u, \qquad r = ux, \qquad \frac{g}{b} = v, \qquad g = vb, \tag{15}$$

we have

$$z = \sqrt{r^2 + x^2} = x\,(1 + u^2)^{\frac{1}{2}}, \qquad y = b\,(1 + v^2)^{\frac{1}{2}}, \tag{16}$$

which, when expanded by the binomial theorem and multiplied, give

$$zy = xb\,(1 + \tfrac{1}{2}\,u^2 + \tfrac{1}{2}\,v^2 - \tfrac{1}{8}\,u^4 + \tfrac{1}{4}\,u^2v^2 - \tfrac{1}{8}\,v^4 \\ + \tfrac{1}{16}\,u^6 - \tfrac{1}{16}\,u^4v^2 - \tfrac{1}{16}\,u^2v^4 + \tfrac{1}{16}\,v^6 + \cdots). \quad (17)$$

The fourth order terms in (17) may be combined into

$$-\tfrac{1}{8}\,(u^2 - v^2)^2$$

and the remaining higher order terms in the series under the symbol

$$\psi\,(u, v),$$

so that

$$zy = xb\,[1 + \tfrac{1}{2}\,u^2 + \tfrac{1}{2}\,v^2 - \tfrac{1}{8}\,(u^2 - v^2)^2 + \psi\,(u, v)]. \quad (18)$$

If this value of zy be substituted in equations (1) and (3), for α and β,

$$\left.\begin{aligned} \alpha &= \sqrt{\tfrac{1}{2}\,rg + \tfrac{1}{4}\,xb\,[u^2 + v^2 - \tfrac{1}{4}\,(u^2 - v^2)^2 + 2\,\psi\,(u, v)]}, \\ \beta &= \sqrt{-\tfrac{1}{2}\,rg + \tfrac{1}{4}\,xb\,[4 + u^2 + v^2 - \tfrac{1}{4}\,(u^2 - v^2)^2 + 2\,\psi\,(u, v)]}. \end{aligned}\right\} \quad (19)$$

The terms

$$xb\,[-\tfrac{1}{4}\,(u^2 - v^2)^2 + 2\,\psi\,(u, v)]$$

vanish at infinite frequencies because they are all of at least the second order in $\frac{1}{\omega}$. Therefore, on substituting for the various remaining quantities their expressions in terms of the fundamental line constants,

$$\left.\begin{aligned} \alpha_\infty &= \alpha\Big]_{f=\infty} = \sqrt{\frac{1}{2}rg + \frac{1}{4}\left(\frac{Cr^2}{L} + \frac{Lg^2}{C}\right)} = \frac{1}{2}\left(r\sqrt{\frac{C}{L}} + g\sqrt{\frac{L}{C}}\right), \\ \beta_\infty &= \beta\Big]_{f=\infty} = \sqrt{\omega^2 LC + \frac{1}{4}\left(\frac{Cr^2}{L} + \frac{Lg^2}{C}\right) - \frac{1}{2}rg} = \omega\sqrt{LC}. \end{aligned}\right\} \quad (20)$$

The curve for α, plotted in terms of the frequency, is asymptotic to the horizontal line at a height given by $\alpha\Big]_{f=\infty}$ in (20). The curve representing β is asymptotic to the straight line passing through the origin, whose equation is $\beta = 2\,\pi f\sqrt{LC}$. As before mentioned, the velocity then approaches $\frac{1}{\sqrt{LC}}$ as a limit.

90. Low-frequency Values of α, β, and *S*. — To investigate the forms of the curves at very low frequencies, the expressions for α and β may be expanded in terms of ω instead of $\frac{1}{\omega}$. Omitting the

details, which are much the same as in the previous case, except that we let

$$x = u_1 r, \quad z = r(1 + u_1^2)^{\frac{1}{2}}, \quad b = v_1 g, \quad y = g(1 + v_1^2)^{\frac{1}{2}},$$

the following series are obtained:

$$\left.\begin{aligned}
\alpha &= \sqrt{rg}\sqrt{1 + \tfrac{1}{4}(u_1^2 + v_1^2) - \tfrac{1}{16}(u_1^2 - v_1^2)^2 + \tfrac{1}{2}\psi(u_1, v_1)} \\
\alpha &= \sqrt{rg}\left(1 + \frac{1}{8}\frac{x^2}{r^2} + \frac{1}{8}\frac{b^2}{g^2} - \cdots\right), \\
\beta &= \sqrt{\tfrac{1}{2}xb + \tfrac{1}{4}rg\left[u_1^2 + v_1^2 - \tfrac{1}{4}(u_1^2 - v_1^2)^2 + 2\psi(u_1, v_1)\right]}, \\
\beta &= \omega\sqrt{\frac{1}{2}LC + \frac{1}{4}\left(\frac{gL^2}{r} + \frac{rC^2}{g}\right) - \frac{1}{16}\omega^2\left(\frac{L^2}{r^2} - \frac{C^2}{g^2}\right)^2 + \cdots}.
\end{aligned}\right\} \quad (21)$$

Thus the limiting velocity of propagation at zero frequency is

$$S_0 = S\Big]_{\omega=0} = \frac{\omega}{\beta} = \frac{1}{\sqrt{\dfrac{1}{2}LC + \dfrac{1}{4}\left(\dfrac{gL^2}{r} + \dfrac{rC^2}{g}\right)}}. \quad (22)$$

It is easily seen that the quantity in parentheses in (22) can never be less than 2 LC, so that the value of β is always greater than $\omega\sqrt{LC}$, except in the special case already mentioned, when $gL = rC$, for which the parenthesis becomes equal to 2 LC. In general, then, the velocity, S, is always less than the infinite frequency velocity.

The two curves representing α and S as functions of the frequency start horizontally from their zero-frequency values, rise with increasing rapidity along curves approximately parabolic, reach their maximum slope at inflexion points where their second derivatives have a value of zero, and then, with continuously decreasing slopes, run out asymptotic to their infinite-frequency values. For lines with low resistance and leakage, the approximately parabolic portions of the curves for low frequencies are very small in comparison with the large sweep of the curves in passing out to their final asymptotic values. With large losses, the curves flatten and approach their final limits more slowly.

91. Illustration of the Variation of α, β, and S with Frequency for Power Circuit. — The following table of values has been computed, to show the manner of variation of these quantities for the power transmission line previously considered, and whose constants per unit length are again given at the beginning of the table.

TABLE XIX

Different-frequency Values of the Attenuation Constant, α, Wave-length Constant, in Degrees, $\beta°$, and the Velocity of Propagation, S, in Miles per Second, for

$L = 0.002040, \quad r = 0.2750, \quad g = 0.15 \times 10^{-6}, \quad C = 0.01460 \times 10^{-6}$

f	α	$\beta°$	S
0	0.00020310	0	94,007
2	0.00022583	0.006888	104,530
4	0.00025820	0.012050	119,510
6	0.00028454	0.016401	131,700
8	0.00030483	0.020412	141,090
10	0.00032070	0.024253	148,440
12	0.00033317	0.028014	154,210
16	0.00035098	0.035456	162,450
25	0.00037205	0.052264	172,200
60	0.00039077	0.11942	180,870
180	0.00039528	0.35418	182,960
300	0.00039566	0.58973	183,140
420	0.00039577	0.82540	183,180
540	0.00039581	1.06111	183,200
660	0.00039583	1.29684	183,210
780	0.00039585	1.53259	183,220
900	0.00039586	1.76833	183,220
2000	0.00039587	3.9294	183,230
3000	0.00039587	5.8941	183,230
4000	0.00039588	7.8588	183,230
6000	0.00039588	11.7882	183,230
8000	0.00039588	15.7176	183,230
Inf.	0.00039588	Inf.	183,230

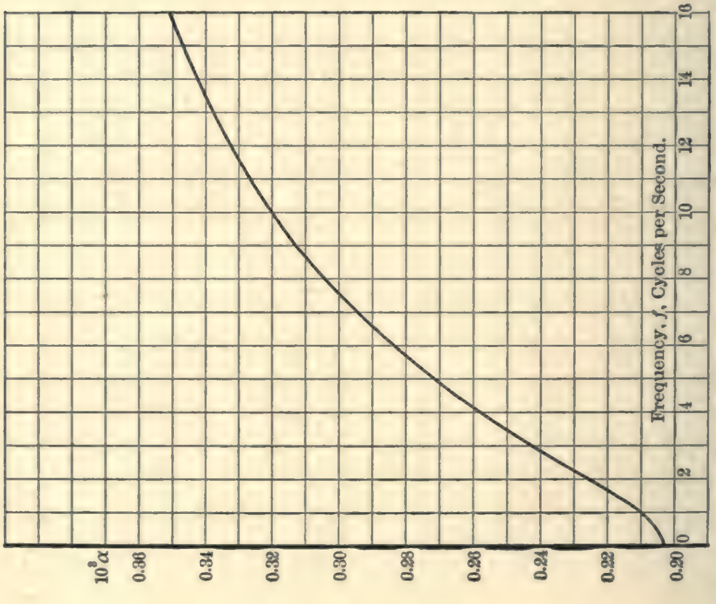

FIG. 49. — The Attenuation Constant α as a Function of the Frequency. (Circuit Constants Representing a Power Line.)

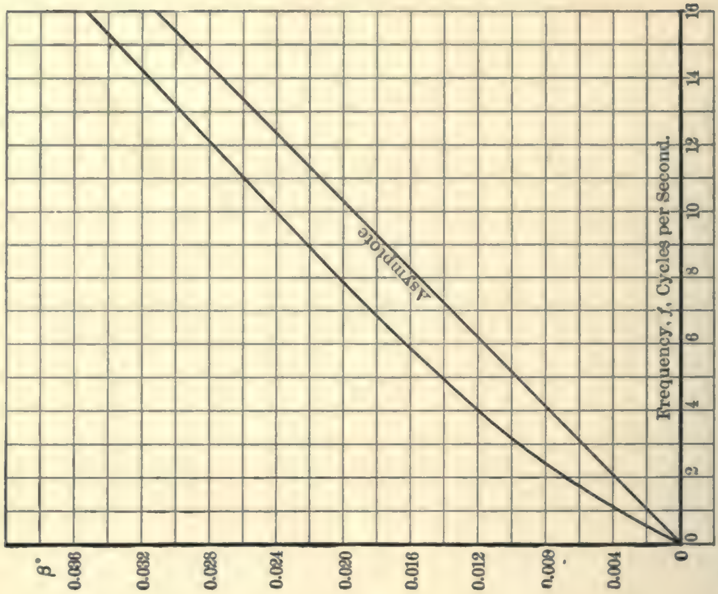

FIG. 50. — The Wave-length Constant β^0 as a Function of the Frequency. (Circuit Constants Representing a Power Line.)

The lower-frequency values entered in Table XIX are plotted as curves in Figs. 49, 50, and 51, in order to show the curve shapes. From an inspection of the table, it is seen that at 60 cycles, α and S have nearly reached their limiting values, and β is approximately

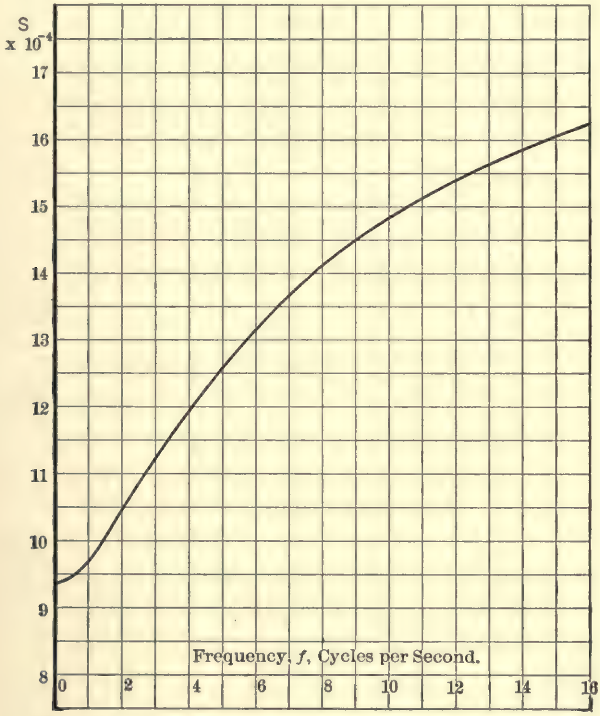

FIG. 51. — The Velocity of Propagation as a Function of the Frequency. (Circuit Constants Representing a Power Line.)

proportional to the frequency. The infinite-frequency values of α and S are nearly twice their zero-frequency values.

92. Relations between α, β, and S. — The similarity between the curve representing α as a function of the frequency and that representing S is so striking that an explanation is needed. In equation (4a), Chapter VI, we have

$$2\,\alpha\beta = rb + gx.$$

This may be written

$$2\,\alpha\beta = \omega\,(rC + gL), \tag{23}$$

from which, in the expression for velocity, we have

$$\left.\begin{aligned} S = \frac{\omega}{\beta} = \frac{2\,\alpha}{rC + gL}, \\ \beta = \frac{rC + gL}{2\,\alpha}\,\omega. \end{aligned}\right\} \tag{24}$$

In the first of equations (24), $rC + gL$ is constant, so that the curves for α and S as functions of the frequency differ only by a constant factor. Thus, knowing any one of the three quantities, α, β, or S, the other two may be quickly determined by means of equation (24) without recourse to the more complicated general expressions for the separate quantities.

93. Numerical Illustrations for Telephone Circuits. — To further illustrate the variation of α, β, and S, the values entered in the following table were computed. The circuit constants used are approximately those for two No. 12, N. B. S. gauge copper wires spaced 12 inches — a standard type of aërial telephone circuit. The values of the constants are for a loop-mile, that is, wire to wire.

TABLE XX

THE PROPAGATION CONSTANT AND VELOCITY OF PROPAGATION IN AN AËRIAL TELEPHONE CIRCUIT OF THE CONSTANTS:

$r = 9.94$, $L = 0.00366$, $g = 1.5 \times 10^{-6}$, $C = 0.00822 \times 10^{-6}$

f	α	$\beta°$	S
0	0.003861	0.00000	88,560
5	0.003874	0.02026	88,850
10	0.003909	0.04015	89,660
20	0.004035	0.07780	92,540
30	0.004203	0.1120	96,400
50	0.004573	0.1716	104,890
100	0.005381	0.2917	123,410
150	0.005967	0.3945	136,870
200	0.006392	0.4911	146,620
300	0.006942	0.6783	159,220
400	0.007260	0.8648	166,520
600	0.007582	1.2423	173,870
800	0.007725	1.6255	177,180
Inf.	0.007949	Inf.	182,310

The above quantities are shown graphically in Figs. 52, 53, 54, and 55.

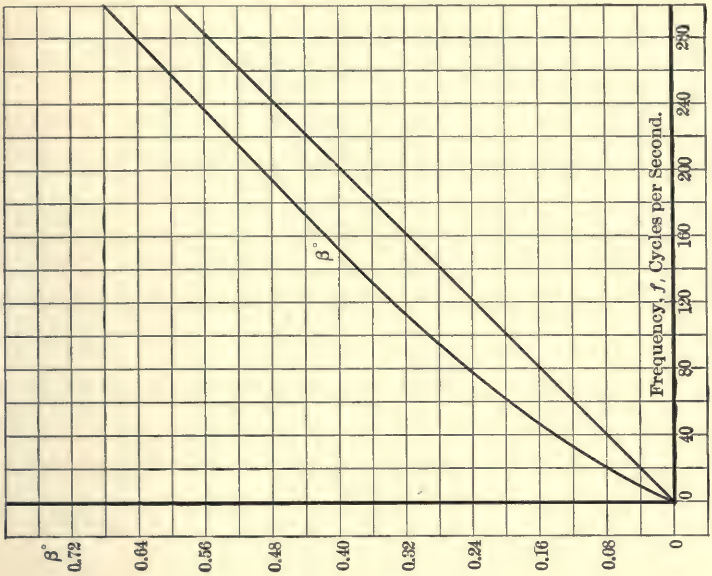

FIG. 53. — The Wave-length Constant for a Telephone Line.

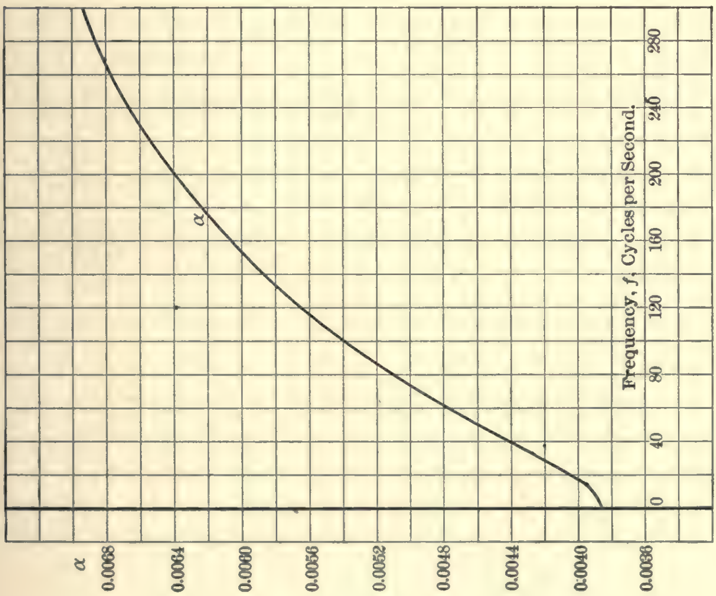

FIG. 52. — The Attenuation Constant for a Telephone Line.

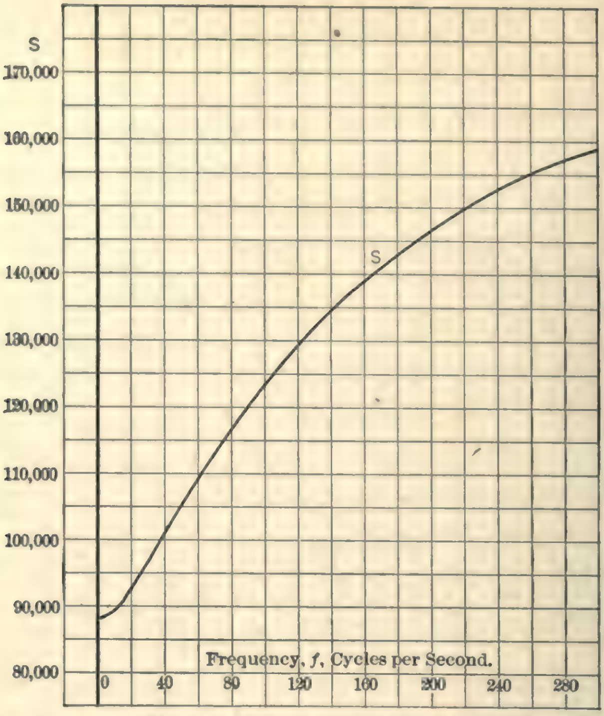

FIG. 54. — The Velocity of Propagation along a Telephone Line.

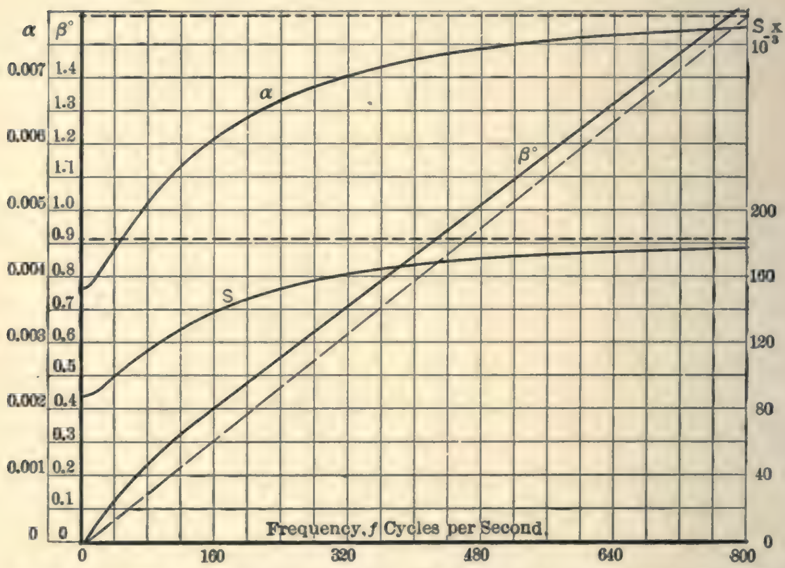

FIG. 55. — The Attenuation Constant, Wave-length Constant, and Velocity of Propagation for a Telephone Line.

CHAPTER VIII

THE EQUIVALENT IMPEDANCE OF CIRCUITS AND THE DETERMINATION OF THE FUNDAMENTAL CONSTANTS BY MEASUREMENTS

94. General Expression for Generator-end Impedance. — The complex ratio between the generator voltage and the generator current, when these two quantities are expressed in complex form from the vector diagram representing them, is the equivalent impedance of the transmission line, including whatever load impedance may be placed at that end of the system. The expression for this impedance is set up in the same manner as for the analogous direct-current problem, viz., by taking the ratio between the general equations in (19), Chapter VII, for E and I in terms of E_0 and I_0. The load-end impedance is the ratio between the load voltage and the load current.

$$Z_0 = \frac{E_0}{I_0} = \text{load-end impedance.} \tag{1}$$

By substituting, $E_0 = Z_0 I_0$ in equations (19), Chapter VII, and dividing, the common factor I_0 canceling from both the numerator and the denominator, we have

$$Z_g = \frac{Z_0 \cosh Vl + U \sinh Vl}{\cosh Vl + \dfrac{Z_0}{U} \sinh Vl}$$

$$= \frac{E}{I} = \text{equivalent generator-end impedance.} \tag{2}$$

In treating the subject of line impedances, it has been the custom of some writers to introduce separate names for the various ratios of e.m.f. to current, and these names are not, in general, such as to be self-explanatory. The introduction of such terms as "Initial sending-end impedance," "Final sending-end impedance," and lastly — the least comprehensible of all — "Final receiving-end impedance," which is not, as would be

supposed, the ratio between load voltage and load current, serve to confuse the mind of the reader. In this discussion we will confine ourselves, in general, to two symbols for equivalent impedance: Z_0 = ratio between load voltage and load current, and Z_g = ratio between generator voltage and generator current. The subscripts used are such as to indicate directly the meaning of the symbol. Both of the impedances, Z_0 and Z_g, are complex, of the form

$$\left.\begin{aligned} Z_0 &= r_0 + jx_0, \\ Z_g &= r_g + jx_g. \end{aligned}\right\} \tag{3}$$

95. Generator-end Impedance under Special Conditions. — Two equations arise from (2) as special cases, which are perhaps more important than the general form. They are the expressions for Z_g when the load end of the line is short-circuited, and when the load end of the line is open-circuited. In the first case, $Z_0 = 0$, and in the second case, $Z_0 = \infty$. Introducing these values of Z_0 and evaluating the resulting equations,

$$\begin{aligned} Z_g\Big]_{Z_0=0} &= Z_g' = U \tanh Vl, \\ &= \text{short-circuited impedance.} \end{aligned} \tag{4}$$

$$\begin{aligned} Z_g\Big]_{Z_0=\infty} &= Z_g'' = U \coth Vl. \\ &= \text{open-circuited impedance.} \end{aligned} \tag{5}$$

These two equations are analogous to equations (39) and (40), in Chapter II, to which they would reduce for $f = 0$.

If, for the purpose of analysis, we assume that the line losses are zero ($r = 0$ and $g = 0$), the equations (4) and (5) assume trigonometric forms because of α being zero. Thus, since

$$\tanh j\beta = j \tan \beta, \qquad \text{and} \qquad \coth j\beta = -j \cot \beta,$$

$$\left.\begin{aligned} Z_g'\Big]_{\alpha=0} &= jU \tan \beta l, \\ Z_g''\Big]_{\alpha=0} &= -jU \cot \beta l. \end{aligned}\right\} \tag{6}$$

Again, when $\alpha = 0$, the quantity U is no longer complex, but

$$U\Big]_{\alpha=0} = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{jx}{jb}} = \sqrt{\frac{L}{C}}, \tag{7}$$

and by equation (6), Chapter VII,

$$\beta\Big]_{\alpha=0} = \omega\sqrt{LC},$$

so, finally, the equivalent generator-end impedances of a line with no losses, the load end being short-circuited and open-circuited, are respectively,

$$\left.\begin{aligned} Z_g'\Big]_{\alpha=0} &= j\sqrt{\frac{L}{C}}\tan\omega\sqrt{LC}l, \\ Z_g''\Big]_{\alpha=0} &= -j\sqrt{\frac{L}{C}}\cot\omega\sqrt{LC}l. \end{aligned}\right\} \quad (8)$$

In such a system it is apparent that, for given values of L and C per unit length, doubling the line length at constant frequency will produce the same variation in Z_g' and Z_g'' as doubling the frequency with the original line length unaltered. This comes about from the fact that ω and l enter the equations (8) as a product, and in no other way. The product $\sqrt{LC}l$ may also be written $\sqrt{L_tC_t}$, where L_t and C_t represent respectively the values of the total line self-inductance and the total line capacity. In (8) the resulting impedances are either pure inductive or pure condensive reactance — no real component appearing in the equations. This must be obvious from physical considerations. If a real component appeared in the expressions for impedance, an energy loss would thereby be represented, which is contrary to the assumption that there is neither resistance nor leakage present.

It is quite obvious that at low frequencies the short-circuited impedance of a line with zero losses will be inductive reactance and approximately equal to ωL_t, numerically. The first of equations (8) reduces to this for low frequencies by neglecting all terms in the series expansion of higher order than the first in ω. It is also obvious that for an open-circuited line at low frequencies the impedance will be condensive reactance and approximately equal in magnitude to $\frac{1}{\omega C_t}$. This is the first term in the series expansion of equation (8), for Z_g''. With increasing frequencies, neither of the two quantities, self-inductance and electrostatic capacity, can be considered alone in such a simple way, but their combined effect must be summed up in equations of the form of those in (8).

96. Analogies between Circuits with Distributed and with Lumped Properties. — It is instructive to compare the results given by (8) for the impedance of circuits having distributed constants with the forms of expression for circuits with lumped constants, which, in an approximate manner, represent physically the short-circuited and open-circuited transmission line. In the case of a short-circuited line we may consider that the current flowing from the generator has, in effect, two alternative paths — one around the entire loop of the line, and thus through the entire amount of self-inductance, and the other through the line electrostatic capacity. The phenomena in this case are approximated by those which take place in a parallel circuit, one path of which contains a condenser and the other an inductance coil. In the

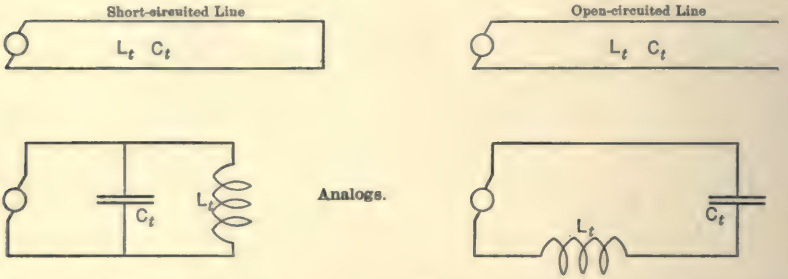

FIG. 56. — Simple Circuits Analogous to Short-circuited and Open-circuited Transmission Lines.

case of an open-circuited line, the analogy is different. Here the generator current may be supposed to flow through two parts of a circuit in series — a portion of the line constituting a self-inductive impedance and the electrostatic capacity of the two wires completing the circuit by a condensive impedance. If the capacity and self-inductance in the localized system be made equal, respectively, to the total capacity and self-inductance of the transmission line, as shown in Fig. 56, the curves giving the reactances of the different circuits appear as in Figs. 57 and 58, which are plotted with frequency as the independent variable for the 400 mile line whose values of L and C are given in Table XII. Thus $L_t = 0.816$ henry and $C_t = 5.84 \times 10^{-6}$ farads.

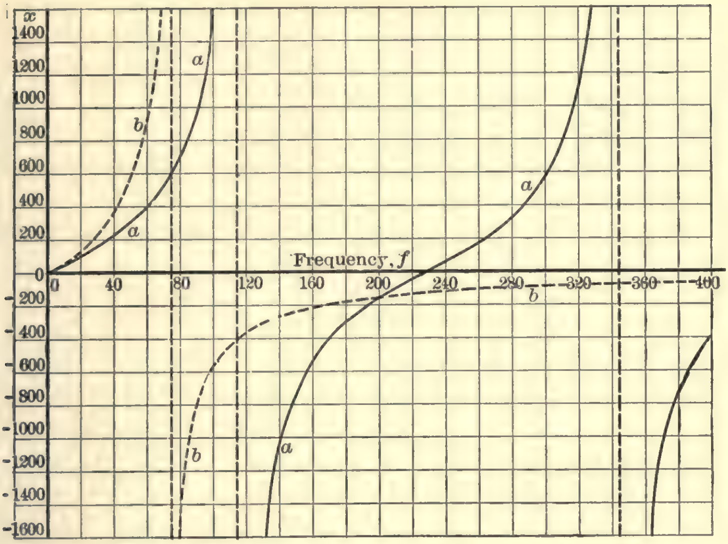

FIG. 57. — The Equivalent Impedance of a Short-circuited Line and of Its Analogous Circuit.

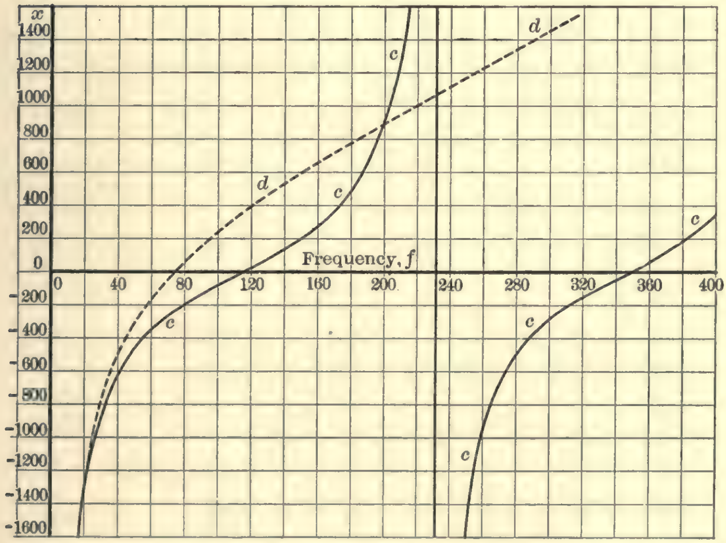

FIG. 58. — The Equivalent Impedance of an Open-circuited Line and of Its Analogous Circuit.

The formulæ for the simple analogous circuits are:

$$\left.\begin{aligned} &\text{(Short-circuited line)} \quad Z = \frac{1}{j\omega C_t - j\,\dfrac{1}{\omega L_t}} = j\,\frac{\omega L_t}{1 - \omega^2 L_t C_t}. \\ &\text{(Open-circuited line)} \quad Z = j\left(\omega L_t - \frac{1}{\omega C_t}\right). \end{aligned}\right\} \quad (9)$$

Numerically, the equations for the actual lines and the analogous circuits are:

1. Short-circuited Line

(a) Line equation, $Z_0'\Big]_{\alpha=0} = j\,373.8 \tan 0.013716 f.$

(b) Analogue, $Z = j\,\dfrac{5.127 f}{1 - 0.00018813 f^2}.$

2. Open-circuited Line

(c) Line equation, $Z_0''\Big]_{\alpha=0} = -j\,373.8 \cot 0.013716 f.$

(d) Analogue, $Z = j\left(5.127 f - \dfrac{1}{0.000035858 f}\right).$

The lettering of the curves in Figs. 57 and 58 corresponds to that in the above numerical equations.

In both comparisons, the two curves in question coincide at zero frequency, but for frequencies above 20 cycles per second they separate rapidly. For the line, the first resonance frequency, as indicated by the infinite value of impedance for the short-circuited line and by the zero impedance for the open-circuited condition, is 114.5 cycles per second. The 400 mile line is exactly one-quarter wave length at this frequency. This value of f may be determined by placing equation (a) equal to infinity. Then $\tan 0.013716 f = \text{inf.}$, or

$$0.013716 f = (2n - 1)\frac{\pi}{2}, \qquad f = \frac{(2n-1)\pi}{0.027432},$$

$$f = (2n - 1)\,114.5.$$

The analogous lumped circuits reach their resonant condition at a much lower frequency than the transmission system — in this case at a frequency of 72.8 cycles per second. The ratio between these

two frequencies, 114.5 and 72.8, is $\frac{\pi}{2}$, and this ratio holds true for any length of line, as may be shown in the following way. The first resonance frequency for the line occurs at a quarter wave length, or when $\beta l = \frac{\pi}{2}$. The resonance frequency in the lumped circuit occurs when the inductive reactance is equal to the condensive reactance, numerically. Thus, for the line at the first resonance condition

$$\beta l = 2\pi f \sqrt{L_t C_t} = \frac{\pi}{2}, \quad f = \frac{1}{4\sqrt{L_t C_t}},$$

and for the lumped circuit

$$2\pi f L_t = \frac{1}{2\pi f C_t}, \quad f = \frac{1}{2\pi \sqrt{L_t C_t}}.$$

The ratio of line frequency to lumped-circuit frequency, at resonance, is thus $\frac{2\pi}{4} = \frac{\pi}{2}$.

In both Fig. 57 and Fig. 58, it is seen that the lumped-circuit impedance is equal to the line impedance at frequencies other than zero — that is, for the frequencies at which the respective curves intersect. In this case, at approximately 200 cycles, the impedances are again equal for both the short-circuited and open-circuited conditions.

The values of the frequencies for equal values of impedance may be determined by equating the respective expressions. For the short-circuited line and its analogous circuit, from equations (8) and (9), for equal impedance,

$$\sqrt{\frac{L_t}{C_t}} \tan \omega \sqrt{L_t C_t} = \frac{\omega L_t}{1 - \omega^2 L_t C_t}, \tag{10}$$

and for the open-circuited line, from the same equations,

$$\sqrt{\frac{L_t}{C_t}} \cot \omega \sqrt{L_t C_t} = \frac{1}{\omega C_t} - \omega L_t. \tag{11}$$

Equation (11) reduces to (10) by taking the reciprocal of both members and then multiplying throughout by $\frac{L_t}{C_t}$, which shows that the impedances in the respective cases become equal at the same frequencies. These transcendental equations, (10) and (11),

are incapable of direct solution for ω, which must, therefore, be determined by methods of approximation or by taking the intersection of curves.

The equations and curves for the equivalent impedances of a line with zero losses and with distributed self-inductance and capacity are of importance in determining the free frequency of oscillation of such circuits, and, in this connection, will be referred to again.

97. Limiting Conditions at Infinite Frequencies. — In actual lines, resistance and some leakage are always present, so that the ideal conditions just considered are never realized exactly. In the actual case, the resulting impedance does not consist of reactance only, which reaches alternately infinite and zero values according to tangent or cotangent trigonometric functions. The energy losses in the line require a real component in the impedance expressions for short-circuited and open-circuited lines, and prevent the reactance curves from reaching infinite values, though they may pass through zero values. The real component of the resulting impedance can, of course, never be negative, for this would imply an average flow of power from the line into the generator, and such a condition is impossible, if the line be either short-circuited or open-circuited as assumed. However they may vary, then, the angles of the two quantities

$$Z_g' = U \tanh Vl \quad \text{and} \quad Z_g'' = U \coth Vl$$

can never attain the limiting values $+90°$ and $-90°$ no matter how long a line or how high a frequency be considered.

Since the attenuation constant, α, approaches a finite limit at very high frequencies, and the wave length constant, β, approaches proportionality with the frequency, it may be shown that, at very high frequencies with a line of fixed length, the curves representing the equivalent resistance and equivalent reactance, for both Z_g' and Z_g'', approach strict periodicity, with frequency as the independent variable. At very high frequencies, the quantity $U = \sqrt{\frac{Z}{Y}}$ approaches the constant $\sqrt{\frac{L}{C}}$ in value. The imaginary portion of U disappears on account of the preponderance of the reactance and susceptance over the resistance and conductance, respectively. By equations (20), Chapter VII, at infinite frequency,

$$\alpha = \alpha_\infty = \text{constant},$$
$$\beta = \omega\sqrt{LC},$$

so that

$$Vl = l\alpha_\infty + j\,2\pi f\sqrt{LC}\,l,$$

and finally,

$$\left.\begin{aligned} Z_0'\Big]_{f=\infty} &= \sqrt{\frac{L}{C}}\tanh\left(l\alpha_\infty + j\,2\pi f\sqrt{LC}\,l\right),\\ Z_0''\Big]_{f=\infty} &= \sqrt{\frac{L}{C}}\coth\left(l\alpha_\infty + j\,2\pi f\sqrt{LC}\,l\right). \end{aligned}\right\} \qquad (12)$$

At these frequencies, the real portion of Vl is constant, and the imaginary portion proportional to f, so that the resulting impedances as given by (12) are periodic in respect to the frequency, with a period of $f_0 = \dfrac{1}{\sqrt{LC}\,l} = \dfrac{1}{\sqrt{L_t C_t}}$, which is the number of cycles increase in frequency required to bring about a recurrence of the same values of impedance.

It will be remembered that, in the case of a line with no losses, $\alpha = 0$, the impedances were both periodic in l with f constant, as well as periodic in f with l constant; and further, periodic in the product fl. The impedance of a line with losses is periodic in f at infinite frequencies only, while with fixed frequency it is never strictly periodic in length as independent variable. If the line length be increased, both components of Vl increase proportionately, so that, though the resulting curves showing the equivalent resistance and equivalent reactance, when plotted in terms of line length, present some of the characteristics of a periodic function, they are not strictly periodic since they do not present recurrently the same sequence of numerical values.

98. Infinite-line Impedance. — If the line be infinitely long, then Vl becomes infinity, and since both the hyperbolic tangent and cotangent approach unity for infinite values of argument (the argument may be either real or complex), the values of open-circuit and short-circuit impedance approach each other for increasing line length, and finally become

$$U = \sqrt{\frac{Z}{Y}} = \text{infinite-line impedance.} \qquad (13)$$

If the attenuation constant α be zero, in which case V is a pure imaginary, the impedances do not approach U in value with increasing line length; for, in such an event, the hyperbolic tangent and cotangent reduce to the trigonometric tangent and cotangent

respectively (neglecting multiplication by $\pm j$), so that for $l = \infty$ the expressions for impedance become indeterminate. In no physical line is it possible for α to be absolutely zero, so that the impedances always approach U = infinite-line impedance, as a limit.

Equation (2), the general expression for equivalent impedance when the line is neither short-circuited nor open-circuited, also reduces to U for infinite values of l — that is, the equivalent generator-end impedance in such a case is independent of the load-end impedance, Z_0.

99. Numerical Illustrations of Equivalent Line Impedances. — For the power transmission line, whose constants are given in Tables XII and XIII, the following numerical values of open-circuited and short-circuited impedances are obtained, by equations (4) and (5) for a line length of 100 miles.

TABLE XXI

SHORT-CIRCUIT AND OPEN-CIRCUIT IMPEDANCES OF A 100 MILE LINE AT DIFFERENT FREQUENCIES

$r = 0.275, \quad L = 0.00204, \quad g = 0.15 \times 10^{-6}, \quad C = 0.0146 \times 10^{-6}$

f	Impedance in ohms	
	Short-circuited	Open-circuited
0	$27.5 + j0$	$66,680.0 - j0$
25	$27.6 + j32.1$	$293.2 - j4331.1$
60	$28.3 + j77.8$	$58.7 - j1789.8$
180	$36.8 + j264.5$	$15.2 - j526.7$
300	$75.8 + j616.4$	$12.7 - j225.2$
420	$865.7 + j2593.9$	$13.9 - j49.2$
540	$165.3 - j1272.0$	$18.0 + j107.5$
660	$29.5 - j449.4$	$29.6 + j309.0$
780	$16.2 - j188.2$	$82.0 + j735.4$
900	$14.6 - j20.8$	$3,255.5 + j4428.1$

The curves in Figs. 59 and 60 are plotted from the above values.

In this case, where the line losses are very small, the equivalent resistance and reactance reach enormous values. For this line, quarter-wave-length phenomena appear at a frequency of 458.06 cycles per second, and half-wave-length phenomena at 916.12 cycles per second. In the short-circuited condition, the impedance

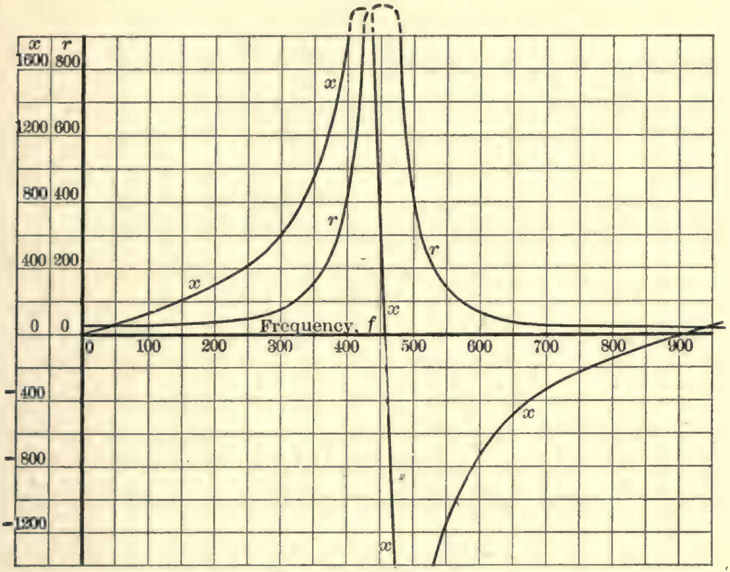

FIG. 59. — The Equivalent Impedance of a Short-circuited, 100 Mile Power Line.

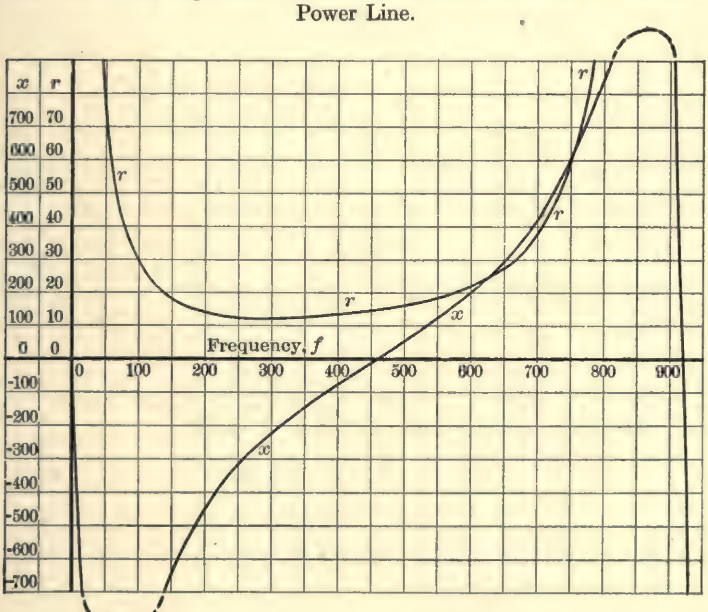

FIG. 60. — The Equivalent Impedance of an Open-circuited, 100 Mile Power Line.

reaches very large values at the frequency which makes the line a quarter-wave length. Beginning at zero frequency, the reactance increases approximately along the tangent curve shown in Fig. 57, but, instead of reversing sign through infinity, reaches a finite maximum value at a frequency slightly less than 458 cycles, drops rapidly through zero and to a negative maximum at a frequency slightly greater than 458 cycles. The numerical values of this negative, or condensive, reactance decrease, then, along a curve approximating the tangent curve, until at a frequency of 916 cycles the half-wave-length phenomena appear and the reactance passes through zero. With continuously increasing frequency, above this value, the curves are sensibly periodic, for, from Table XIX, α has become practically constant. It is interesting to note that, at the frequency corresponding to half-wave-length phenomena, the effective generator-end resistance is only a little more than one-half its zero-frequency value.

The open-circuited impedance is, of course, very high at low frequencies. The equivalent resistance at zero frequency is 66,680 ohms — slightly greater than the reciprocal of the total line conductance,

$$\frac{10^6}{100 \times 0.150} = 66{,}667 \text{ ohms.}$$

The equivalent reactance at zero frequency is zero, but it rises rapidly to very high numerical values, approximating the reciprocal of the total line susceptance. The numerical value of this negative, condensive, reactance decreases along a curve approximating the cotangent curve shown in Fig. 58, passes through zero at a frequency of 458 cycles per second, where the quarter-wave-length phenomena appear, and then continues approximately periodic in frequency as independent variable. At a frequency of 300 cycles, the equivalent resistance is less than one-half the total line resistance.

When longer lines or lines with larger values of r and g are considered, the equivalent resistance and reactance do not pass through such extended ranges of values. As a comparison with the above case, the values of the short-circuited and open-circuited impedances of the No. 12 N. B. S. gauge telephone circuit, for which the values of α and β are given in Table XX, are tabulated. The length of line is taken as 300 miles.

TABLE XXII

SHORT-CIRCUITED AND OPEN-CIRCUITED IMPEDANCES OF 300 MILE TELEPHONE LINE

$r = 9.94$, $L = 0.00366$, $g = 1.5 \times 10^{-6}$, $C = 0.00822 \times 10^{-6}$

f	Impedance	
	Short-circuited	Open-circuited
0	$2112 - j\,0$	$3137 - j\,0$
5	$2108 - j\,80$	$3074 - j\,374$
10	$2095 - j\,158$	$2902 - j\,689$
20	$2047 - j\,307$	$2423 - j\,1047$
30	$1973 - j\,437$	$1991 - j\,1125$
50	$1776 - j\,627$	$1478 - j\,989$
100	$1291 - j\,745$	$1093 - j\,651$
150	$1003 - j\,642$	$995 - j\,513$
200	$865 - j\,514$	$936 - j\,461$
300	$791 - j\,345$	$808 - j\,397$
400	$773 - j\,297$	$732 - j\,297$
500	$720 - j\,263$	$732 - j\,230$
600	$698 - j\,200$	$722 - j\,221$
700	$713 - j\,177$	$686 - j\,189$
800	$693 - j\,176$	$692 - j\,148$

As may be seen by an inspection of the values of impedance in Table XXII, a line of this length and with these values for the fundamental constants, has no inductive reactance at low frequencies even when short-circuited. For the range of frequencies covered in Figs. 61 and 62, plotted from the above table, the reactance is condensive throughout — both when short-circuited and when open-circuited. At very high frequencies, the equivalent reactances decrease, numerically, and finally oscillate about a zero value, while the equivalent resistances oscillate about some finite limiting value.

A comparison of Figs. 61 and 62 with Figs. 59 and 60 shows strikingly the effect of an increase in the values of line resistance and leakage, and therewith an increase in the attenuation constant α.

100. Determination of Line Constants from Short-circuit and Open-circuit Tests. — It is frequently necessary to determine, by actual measurement, the effective values of the four funda-

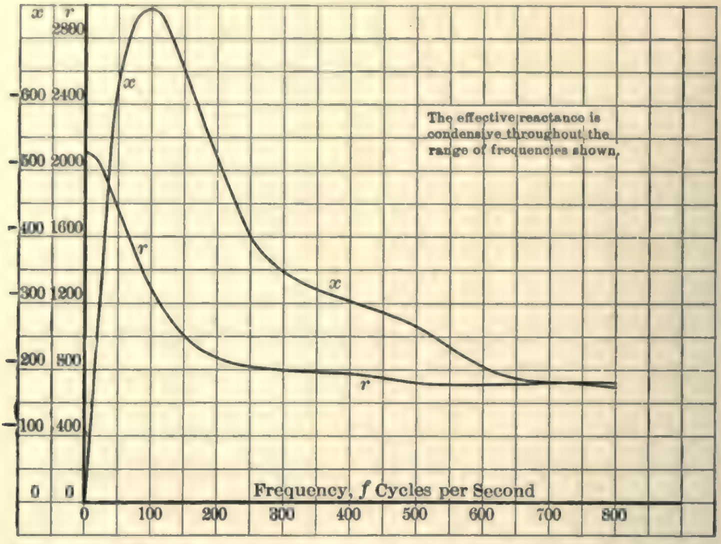

FIG. 61. — The Equivalent Impedance of a Short-circuited, 300 Mile Telephone Line.

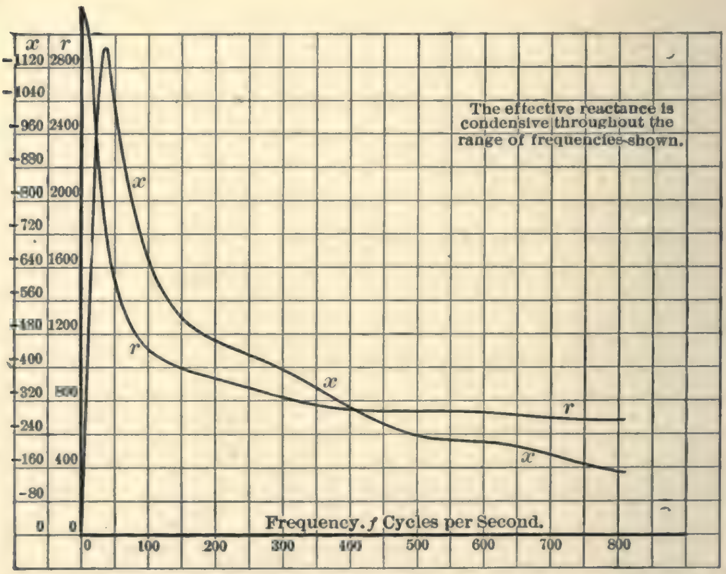

FIG. 62. — The Equivalent Impedance of an Open-circuited, 300 Mile Telephone Line.

mental line constants at some particular frequency. On account of the variation of these constants with changes of frequency, it is, in general, not possible to measure the resistance and leakage of the line with direct current, nor to measure the capacity and self-inductance by any ballistic galvanometer method. If alternating current be used to make such determinations, all of the line constants come into play, so that any one measurement will not serve to determine the four constants. The frequency used in the measurement must be the same as that for which the values of r, L, g, and C are desired. Of course, in some cases, where approximate results only are desired, such simple procedures as the determination of the line capacity from an open-circuit test by the equation

$$C_t = \frac{I_g}{2\pi f E_g} \text{ farads,}$$

where E_g and I_g are respectively the generator voltage and generator current when the line is open-circuited, may suffice. However, if the line is electrically long — that is, approaches a quarter wave-length — or if the resistance or leakage coefficients are large, such a procedure may not yield even an approximate result. Accurate values of the constants may be obtained by using two measurements — one of the short-circuit impedance and one of the open-circuit impedance. When both components of the impedances are measured, four numerical values are obtained as data from which the values of the four line constants may be calculated.

Suppose that, by measurement at the particular frequency for which the effective values of the line constants are desired, actual numerical values are obtained for

$$Z_g' = \text{short-circuit impedance}$$

and

$$Z_g'' = \text{open-circuit impedance,}$$

the line length, l, being known. The unknown quantities are Z and Y per unit length, or U and V, which are expressed in terms of Z and Y by equations (4) and (6), Chapter VI, and from which Z and Y may be obtained.

Now

$$Z_g' = U \tanh Vl$$

and

$$Z_g'' = U \coth Vl,$$

so that by multiplication

$$\left.\begin{aligned} Z_g'Z_g'' &= U^2, \\ U &= \sqrt{Z_g'Z_g''}. \end{aligned}\right\} \tag{14}$$

Further,

$$\tanh Vl = \frac{\sinh Vl}{\cosh Vl} = \frac{\epsilon^{Vl} - \epsilon^{-Vl}}{\epsilon^{Vl} + \epsilon^{-Vl}} = \frac{1 - \epsilon^{-2Vl}}{1 + \epsilon^{-2Vl}},$$

so that

$$Z_g' = U \frac{1 - \epsilon^{-2Vl}}{1 + \epsilon^{-2Vl}}, \tag{15}$$

which, on solving for $\epsilon^{2Vl} = \frac{1}{\epsilon^{-2Vl}}$, gives

$$\epsilon^{2Vl} = \frac{U + Z_g'}{U - Z_g'},$$

or

$$V = \frac{1}{2l} \ln \frac{U + Z_g'}{U - Z_g'} = \frac{1}{2l} \ln (A + jB), \tag{16}$$

where the notation, $\ln x$, signifies "the logarithm to base ϵ of x."

The quantity $\frac{U + Z_g'}{U - Z_g'}$, is complex, of the form $A + jB$, and it is of this quantity that the logarithm is required. Since

$$\epsilon^{u+jv} = \epsilon^u (\cos v + j \sin v) = \epsilon^u \underline{/v \text{ (radians)}},$$

we have

$$u + jv = \ln \epsilon^u (\cos v + j \sin v) = \ln (\epsilon^u \underline{/v}).$$

Therefore,

$$\left.\begin{aligned} \ln (A + jB) &= \ln \left(\sqrt{A^2 + B^2}\underline{/\theta}\right) \\ &= \ln \sqrt{A^2 + B^2} + j\theta, \\ \text{where} \qquad \theta &= \arctan \frac{B}{A}. \end{aligned}\right\} \tag{17}$$

Thus

$$V = \frac{1}{2l} \ln \sqrt{A^2 + B^2} + j \frac{1}{2l} \arctan \frac{B}{A}. \tag{18}$$

With U and V from the preceding equations,

$$U = \sqrt{\frac{Z}{Y}} \qquad \text{and} \qquad V = \sqrt{ZY},$$

$$Z = r + j\omega L = UV \qquad \text{and} \qquad Y = g + j\omega C = \frac{V}{U}, \tag{19}$$

which thus determines the four line constants.

101. Multiple Solutions. — In using equation (18) it is impossible to avoid a certain ambiguity. In taking the arc tan $\frac{B}{A}$, any multiple of 2π may be added to the result at will, thus permitting of a variety of values for the imaginary portion of V. Multiplying (18) by l,

$$Vl = \alpha l + j\beta l = \frac{1}{2}\ln\sqrt{A^2+B^2} + j\,\frac{1}{2}\text{arc tan}\,\frac{B}{A}.$$

$$\left.\begin{aligned} \alpha l &= \frac{1}{2}\ln\sqrt{A^2+B^2}, \\ \beta l &= \frac{1}{2}\text{arc tan}\,\frac{B}{A}. \end{aligned}\right\} \qquad (20)$$

When the line length and frequency are not too great and the conductors are in air, an estimate as to the probable value of βl may be made in order to decide whether or not a multiple of 2π is to be added to arc tan $\frac{B}{A}$. In (20), βl, itself, may differ from the true value by any multiple of π, since one-half the arc tan $\frac{B}{A}$ is involved. If the measurement be made upon a circuit consisting of a long cable, where it would be impossible to estimate accurately the number of wave lengths involved, it would be necessary to commence the measurements at frequencies for which the cable would certainly be less than one-half wave length, and then, by taking measurements at a number of increased frequencies, to plot the impedance curves in order to decide upon the number of times π must be added to $\frac{1}{2}$ arc tan $\frac{B}{A}$ in determining βl by (20). The quantity α is obtained without any difficulties of this kind. When the line is very long, the differences between the measured impedances and U will be very small, since U is the infinite-line impedance, so that considerable inaccuracy may result from equation (16) on account of the small difference between two nearly equal quantities in the denominator.

102. Methods of Making Impedance Measurements. — The impedance measurements may be made by the voltmeter-ammeter-wattmeter method, or by means of an alternating current bridge, depending upon the amount of power available and upon the character of the transmission line under investigation. When the

first method is used, in the case of a power line with large conductors, it will probably be difficult to obtain the real component of the impedance accurately on account of the very low power-factors— particularly with the line open-circuited. When an alternating current bridge is used, the current and voltage impressed upon the line will probably be much smaller in value than those obtaining when the line is in actual use. However, as long as no corona formation occurs when the line is in use for its regular purpose, and no ironclad circuits are present, the values of the constants obtained by a bridge method with small amounts of power would not be subject to change when greater voltage and current are used. The voltmeter-ammeter-wattmeter method of measuring both components of impedances is so familiar as not to need description.

103. An Impedance Bridge. — A form of alternating current bridge which the writer has found to be very satisfactory is shown in Fig. 63.

In the two upper figures, a and b, the bridges are shown diagrammatically, and in the lower figure are shown the connections to a double-throw, double-pole switch to be used for changing from scheme a to scheme b. R_1, R_2, and R are adjustable non-inductive resistances, and C_0 is an adjustable standard capacity. Connection "a" gives directly the two components of a condensive admittance, $Y = g + j\omega C$, so that it is necessary to take the reciprocal of this value to obtain the impedance. Connection "b" gives the two components of an inductive impedance, $Z = r + j\omega L$. The formulæ to be used in the reductions are given beneath the diagrams, and in them the values of R, R_1, R_2, and C_0 which produce a balanced condition in the bridge are to be substituted. To detect a condition of balance, either a telephone receiver or a vibration galvanometer may be used — the latter being, in many instances, the more satisfactory. When a telephone receiver is used, frequencies much below 250 cycles per second cannot be employed with the bridge on account of the inaudibility of their tone, and again, with any circuit as unknown whose apparent constants change with the frequency, a balance at one frequency will not be a balance at another frequency, so that with any ordinary source of e.m.f. difficulty will arise from the presence of higher harmonics in the e.m.f. wave. The sound produced by them will always be present and serve to mask the disappearance

of the fundamental tone at a condition of balance. The vibration galvanometer, on the other hand, may be used at low frequencies as well as at frequencies of hundreds of cycles per second, and since the period of the instrument is adjusted to coincide with the period of the e.m.f. used, no trouble will arise from the presence of higher harmonics of ordinary magnitudes — say a few per cent.

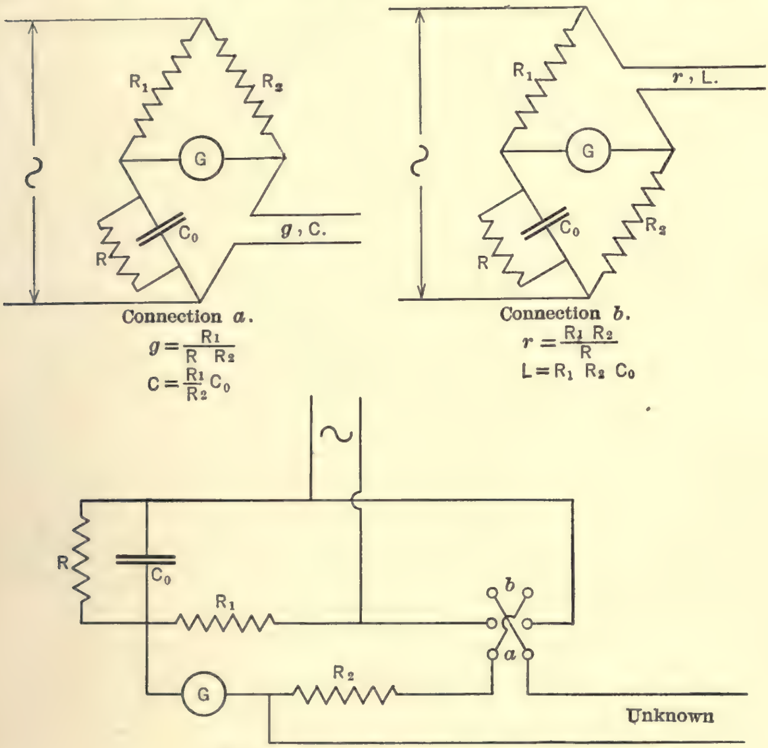

FIG. 63. — An Impedance Bridge.

For R_1 and R_2, resistances with several steps and with a total value of about 2000 ohms each are appropriate, except for the measurement of small inductances, by connection "b," where values as low as 50 or 100 ohms may profitably be employed.

104. Precautions in the Use of the Bridge. — To avoid excessive values of R it is often desirable to shunt the unknown with a known non-inductive resistance when measuring condensive admittance by connection "a," or to insert a known non-inductive resistance in series with the unknown when measuring inductive

reactance by connection "b." The values of g or r thus introduced are to be subtracted from the measurements. The alternating e.m.f. supplied to the bridge should be taken from the secondary of an air-core or other insulating transformer placed at a distance sufficient to avoid the influence of extraneous magnetic fields. This insulating transformer is needed to prevent the introduction of errors on account of the large electrostatic capacity between the generator windings and leads and earth. If great accuracy is desired, the different portions of the bridge should be electrostatically shielded, and, in any case, care should be taken so to distribute the different parts of the apparatus that the mutual interactions be reduced to a minimum.

Considerable difficulty is often experienced in finding the proper adjustment for a balance. When measurements of a transmission line are undertaken, one usually has no knowledge of the approximate values, even, of the unknowns, so that a balance must be sought by trial. When a long line is involved and the frequency is high, it cannot be known beforehand whether the unknown impedance is inductive or condensive. Trial must then be made of both bridge connections. Since there are two quantities which require simultaneous adjustment, C_0 and R, the best that can be done is to first adjust one of them until the current in the galvanometer is a minimum, then adjust the other until a new minimum is obtained, and so on. A little experience enables these adjustments to be made very rapidly after a condition is found where a change in either C_0 or R produces a noticeable change in the galvanometer current. Care must be taken to see that the frequency used does not approach those for which the impedances under measurement vary rapidly, for then small variations in the frequency of the e.m.f. supplied to the bridge would cause the bridge settings for a balance to vary greatly, and thus prevent any accurate adjustments. For example, in an impedance measurement of the short-circuited 100 mile line, whose effective resistance and reactance are shown as a function of the frequency in Fig. 59, reliable measurements would be difficult with frequencies between 350 and 500 cycles per second. According to Fig. 60, for this same line open-circuited, measurements would be difficult with frequencies between 800 and 1000 cycles per second, as well as at 450 cycles, where the impedance would change from inductive to condensive, or vice versa, for small changes of frequency.

105. Illustrative Computation. — To illustrate the use of equations (14) to (19), consider the following numerical example. On a given line, 41 miles in length, impedance measurements with the following results were made at a frequency of 660 cycles per second:

Line short-circuited. Connection "b"	Line open-circuited. Connection "a"
$R_1 = 1000 \quad R_2 = 1000$ $R = 1224.0 \quad C_0 = 0.2766 \times 10^{-6}$	$R_1 = 1000 \quad R_2 = 1000$ $R = 816.7 \quad C_0 = 0.7816 \times 10^{-6}$

By using the respective formulæ in the two cases, "a" and "b": Short-circuit test, "b,"

$$Z_g' = 817 + j\,1147 \text{ ohms.}$$

Open-circuit test, "a,"

$$Y_g'' = 0.0012244 + j\,0.0032411 \text{ mhos.}$$

$$\frac{1}{Y_g''} = Z_g'' = 102.0 - j\,270.0 \text{ ohms.}$$

These values are to be used as original data with which to enter equation (14). The computations are most easily done with logarithms, according to the following table, where in place of the quantity itself, its logarithm is given, unless otherwise stated. ϕ represents, in general, the angle of a complex quantity; for example,

$$Z = r + jx = z\underline{/\phi}.$$

These are the effective values of the line constants at this frequency (660 cycles per second) per mile of length, and they will, in general, be different for different frequencies.

106. Effect of Errors in Original Data. — In this numerical example, the fundamental data, Z_g' and Z_g'', are such as to yield very accurate results, for the two impedances as measured are sufficiently different from each other to permit an accurate formation of their sums and differences, as required in equation (16). Further, the resulting values of impedance and admittance per unit length of line have angles which are not very nearly equal to either 0 or 90 degrees. The determination of the quantity g is usually the most unsatisfactory, for the angle of the admittance, Y, is frequently very near 90 degrees, so that g, the real component,

TABLE XXIII

Calculation of Line Constants from Two Impedance Measurements. Equations 14 to 20

Z_g'	$2.91222 + j\,3.05956$	$\tan\phi'$	0.14734
$\sin\phi'$	9.91089		
Z_g''	$2.00860 - j\,2.43136$	$\tan\phi''$	$-$ 0.42276
$\sin\phi''$	$-$ 9.97103		
Z_g'	$3.14867\underline{/54.538 \text{ deg.}}$		
Z_g''	$2.46033\underline{/-69.305 \text{ deg.}}$		
$Z_g'Z_g''$	$5.60900\underline{/-14.767 \text{ deg.}}$		
$\cos\phi_u$	9.99638–10		
U	$2.80450\underline{/-7.3835 \text{ deg.}}$		
$\sin\phi_u$	$-$ 9.10893–10		
U	$2.80088 - j\,1.91343$		
U	$632.25 - j\,81.93$	Numerically	
Z_g'	$817.00 + j\,1147.00$	Numerically	

Let $m = U + Z_g'$ and $n = U - Z_g'$

$m = U + Z_g'$	$1449.25 + j\,1065.07$	Numerically	
$n = U - Z_g'$	$-184.75 - j\,1228.93$	Numerically	
m	$3.16114 + j\,3.02738$	$\tan\phi_m$	9.86624–10
$\cos\phi_m$	9.90623–10		
n	$2.26659 - j\,3.08953$	$\tan\phi_n$	$-$ 0.82294
$\sin\phi_n$	$-$ 9.99515		
m	$3.25491\underline{/36.313 \text{ deg.}}$		
n	$3.09438\underline{/261.450 \text{ deg.}}$		

$A + jB = \dfrac{m}{n}$ $0.16053\underline{/134.863 \text{ deg.}}$ (Logarithm of scalar)

$\log\log\dfrac{m}{n}$	9.20557–10	$\phi° = \tan^{-1}\dfrac{B}{A}$	134.863 deg.
log Mod.	9.63778–10	$\log\phi°$	2.12989
$\log\ln\dfrac{m}{n}$	9.56779–10	$\log\dfrac{360}{2\pi}$	1.75812
$\log 2\,l$	1.91381	$\log \text{arc} \tan\dfrac{B}{A}$	0.37177
		$\log 2\,l$	1.91381

	$7.65398\text{–}10 + j\,8.45796\text{–}10$	$\tan\phi_v$	0.80398
$\sin\phi_v$	9.99471–10		
V	$8.46325\text{–}10\underline{/81.075 \text{ deg.}}$		
U	$2.80450\ \underline{/-7.383 \text{ deg.}}$		
ϕ_z	73.692 deg.	ϕ_y	88.458 deg.
$\cos\phi_z$	9.44841–10	$\cos\phi_y$	8.42985–10
z	1.26775	y	5.65875–10
$\sin\phi_z$	9.98216–10	$\sin\phi_y$	9.99984–10
r	0.71616	g	4.08860–10
x	1.24991	b	5.65859–10
$2\pi f$	3.61772	$2\pi f$	3.61772
L	7.63219–10	C	2.04087–10

Final numerical results

$r = 5.2018$, $L = 0.0042873$, $g = 1.2263 \times 10^{-6}$ $C = 0.010987 \times 10^{-6}$

is not accurately determined. In the particular case just treated, the angle of Y is 88.458 degrees — only 1.542 degrees different from 90 degrees. In general, the error in the angle of Y is of a lower order of magnitude than the original errors in the measured angles of the equivalent impedances, Z_g' and Z_g'', so that in this case, the cosine of the angle 88.458 degrees, on which the determination of g depends, is still obtained with considerable accuracy.

To illustrate the effects of an error in the original measurements, suppose that, in determining Z_g', an error of one degree is made by a wrong determination of the imaginary component of the impedance — the real component being assumed correct. Then as original data, in logarithms,

$$Z_g' = 2.91222 + j\,3.07570 = 3.15951\underline{/55.538\text{ deg.}}$$

$$Z_g'' = \text{same as before.}$$

From this fundamental data,

$$r = 5.0394, \qquad L = 0.0043677, \qquad g = 1.4026 \times 10^{-6},$$
$$C = 0.010874 \times 10^{-6}.$$

In this latter case, the angle of Y is 88.219 degrees, as compared with 88.458 degrees in the previous case based on the correct values of Z_g' and Z_g''. The change in the resulting angle of Y due to a change of one degree in the angle of Z_g' is 0.239 degrees.

As mentioned before, the best results from measurements of this kind are obtained under conditions which cause the two impedances, Z_g' and Z_g'', to be considerably different, for then the value of the fraction in equation (16) is obtained without loss of accuracy. The shorter the line under test, the greater the difference in these impedances. With very short lines, however, when the resistance and leakage coefficients are small, it will be difficult to obtain accurate values of the real components of the effective impedances by measurement, so that, though the subsequent numerical solution will be easy to carry out accurately, the errors in the fundamental data may be serious. With the 300 mile telephone circuit, whose impedances are tabulated in Table XXII, for example, values of line constants based on impedance measurements at frequencies above 100 cycles would be quite inaccurate, on account of the very nearly equal values of Z_g' and Z_g''. If the effective line constants were desired at higher frequencies, it would

be desirable, or even necessary, to make the impedance measurements on a shorter section of the line — say 100 or 50 miles, depending on the frequency. On the other hand, measurements on a 300 mile power transmission line, whose values of r and g are usually much smaller than for the telephone circuit, would probably furnish very satisfactory data.

CHAPTER IX

CHANGE OF WAVE SHAPE ALONG LINES

107. Method of Treatment of Complex Wave Shapes. — The equations given in the foregoing chapter are all based on the complex quantity representation of alternating quantities, and thus yield solutions for a single frequency. By the definition of an ideal line, however, in which it is stated that the effective values of the line constants in such an ideal line are independent of the magnitudes of the voltage or current, the treatment of complex wave shapes by the superposition of separate solutions for the separate frequencies is made possible. If the e.m.f. wave of a generator which supplies a particular line contains a number of higher harmonics, the voltage wave shape at any point in the line may be found by determining the magnitude and phase position of the several component voltages at the point in question by applying the complex quantity method of analysis to each separately, and then taking the instantaneous sums as data from which to plot the resultant curve — if the curve, and not merely the values of the separate components, be needed. The method of treatment is essentially the same as used in the correction of oscillographic curves of wave shape for the effect of distortion circuits, Chapter IV. If any of the generator e.m.f. harmonics should coincide in frequency with a resonance frequency of the line, and the line losses be small, very great exaggeration of this particular harmonic, either in the voltage or in the current produced thereby, may take place in portions of the line. For a consideration of such phenomena, it will be necessary to reproduce, in the form for use in alternating current systems, some of the specialized equations given in Chapter II for leaky direct current lines.

108. General Case; Load Impedance = Z_0. — The fundamental equations are:

$$\left.\begin{aligned} E &= E_0 \cosh Vl + I_0 U \sinh Vl, \\ I &= I_0 \cosh Vl + E_0 \frac{1}{U} \sinh Vl, \end{aligned}\right\} \qquad (1)$$

which give the voltage and current at any point distant l from the load-end of the line in terms of the load-end voltage and current. Solving simultaneously for E_0 and I_0 (see eq. 16, Chap. II),

$$\left.\begin{aligned} E_0 &= E \cosh Vl - IU \sinh Vl, \\ I_0 &= I \cosh Vl - E \frac{1}{U} \sinh Vl, \end{aligned}\right\} \qquad (2)$$

which give the voltage and current at any point distant l from the generator in terms of the generator voltage and current, E and I.

To determine the voltage and current at any point in a given line when the load-end impedance, Z_0, is known, let

l_0 = total line length between generator and load, Z_0,

and let E' and I' represent the voltage and current respectively at any point distant l' from the generator, whose voltage is E. l' is thus always less than l_0.

The ratio between current and voltage at the generator is the equivalent generator-end impedance, which, by equation (2), Chapter VIII, is

$$\left.\begin{aligned} Z_g &= \frac{Z_0 \cosh Vl_0 + U \sinh Vl_0}{\cosh Vl_0 + \frac{Z_0}{U} \sinh Vl_0}, \\ Y_g &= \frac{1}{Z_g} \quad \text{and} \quad I = EY_g. \end{aligned}\right\} \qquad (3)$$

Therefore, applying equation (2), for a distance l' from the generator and eliminating I by equation (3),

$$\left.\begin{aligned} E' &= E\left(\cosh Vl' - UY_g \sinh Vl'\right), \\ I' &= E\left(Y_g \cosh Vl' - \frac{1}{U} \sinh Vl'\right). \end{aligned}\right\} \qquad (4)$$

In computing from this equation, first determine the value of Y_g and then substitute it in the equation in connection with the particular values of l'. No benefit is to be derived from the substitution of the expression in (3) for Y_g in equation (4), except numerically, as mentioned.

109. Short-circuited and Open-circuited Line. — Since considerable labor is required to determine the values of Y_g, in general, the two special cases — short-circuited line and open-circuited line — are of importance, for then the values of Z_g or Y_g are easily

obtained. For these special cases, the substitution of the expressions for the equivalent generator-end impedance from equations (4) and (5), Chapter VIII, gives

Short-circuited:

$$\left.\begin{aligned} E'\Big]_{Z_0=0} &= E(\cosh Vl' - \coth Vl_0 \sinh Vl') = E\frac{\sinh V(l_0 - l')}{\sinh Vl_0}, \\ I'\Big]_{Z_0=0} &= \frac{E}{U}(\coth Vl_0 \cosh Vl' - \sinh Vl') = E\frac{\cosh V(l_0 - l')}{U \sinh Vl_0}. \end{aligned}\right\} \quad (5)$$

Open-circuited:

$$\left.\begin{aligned} E'\Big]_{Z_0=\infty} &= E(\cosh Vl' - \tanh Vl_0 \sinh Vl') = E\frac{\cosh V(l_0 - l')}{\cosh Vl_0}, \\ I'\Big]_{Z_0=\infty} &= \frac{E}{U}(\tanh Vl_0 \cosh Vl' - \sinh Vl') = E\frac{\sinh V(l_0 - l')}{U \cosh Vl_0}. \end{aligned}\right\} \quad (6)$$

If we make $l' = l_0$, then equation (4) must reduce to forms similar to equations (19) and (20), Chapter II — giving the load voltage and current in terms of the load impedance, generator voltage, and total line length.

Making the substitution, $l' = l_0$, and reducing,

$$\left.\begin{aligned} E_0 &= \frac{E}{\cosh Vl_0 + Y_0 U \sinh Vl_0}, \\ I_0 &= \frac{E}{Z_0 \cosh Vl_0 + U \sinh Vl_0}. \end{aligned}\right\} \quad (7)$$

From these, for the two special cases,

$$\left.\begin{aligned} E_0\Big]_{Z_0=\infty} &= E \operatorname{sech} Vl_0, \\ I_0\Big]_{Z_0=0} &= E\,\frac{1}{U} \operatorname{cosech} Vl_0. \end{aligned}\right\} \quad (8)$$

In a direct current system, the quantity V is always real, and thus in equation (8), for instance, the quantity $E \operatorname{sech} Vl_0$ continuously decreases for increasing line length. In the alternating current system, V is complex, and a function of the frequency, so that the value of $\operatorname{sech} Vl_0$ may vary through very great numerical ranges, and in the case of a line with no losses in which the attenuation constant is zero, throughout the range from $-\infty$ to $+\infty$. Similarly with $\operatorname{cosech} Vl_0$. The voltages at the terminal of an open-circuited line may thus rise to very high values, as well as

the current at the end of a short-circuited line. Equations (5) and (6) indicate that excessive voltages or currents may also occur at intermediate points in the line length, if the total line length, l_0, be such as to give large values to coth Vl_0 or to tanh Vl_0, depending upon whether the line is short-circuited or open-circuited. If the line be connected to an impedance Z_0, the excessive currents or voltages may or may not occur, depending on the resulting value of Y_g, from equation (3). Y_g becomes very large when

$$Z_0 \cosh Vl_0 + U \sinh Vl_0 \text{ approaches zero.}$$

For example, in a line with no losses ($\alpha = 0$),

$$\cosh Vl_0 = \cos \beta l_0,$$
$$\sinh Vl_0 = j \sin \beta l_0,$$
$$U = \sqrt{\frac{L}{C}},$$

so that

$$Y_g = \infty \qquad \text{when} \qquad Z_0 \cos \beta l_0 + j\sqrt{\frac{L}{C}} \sin \beta l_0 = 0,$$

or when

$$Z_0 = -j\sqrt{\frac{L}{C}} \tan \beta l_0.$$

Infinite values of Y_g, however, imply an equivalent short-circuit on the generator supplying the line, and under such a condition it is not reasonable to suppose the generator voltage to remain constant, for then we would have infinite currents throughout the line, as well as infinite voltages. In such cases — very large values of Y_g — the phenomena must be expressed in terms of the generator current I, rather than the generator voltage, E, the former being more or less definite at short-circuit, while the latter disappears. The equations in terms of I are obtained by substituting IZ_g for E, and then, from equations (4), (5), (6), (7), and (8), we have

$$\left.\begin{aligned} E' &= I\,(Z_g \cosh Vl' - U \sinh Vl'), \\ I' &= I\left(\cosh Vl' - \frac{Z_g}{U} \sinh Vl'\right). \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} E'\Big]_{Z_0=0} &= IU(\tanh Vl_0 \cosh Vl' - \sinh Vl') = IU\,\frac{\sinh V(l_0 - l')}{\cosh Vl_0}, \\ I'\Big]_{Z_0=0} &= I\,(\cosh Vl' - \tanh Vl_0 \sinh Vl') = I\,\frac{\cosh V(l_0 - l')}{\cosh Vl_0}. \end{aligned}\right\} \tag{10}$$

$$
\left.\begin{aligned}
E'\Big]_{Z_0=\infty} &= IU(\coth Vl_0 \cosh Vl' - \sinh Vl') = IU\frac{\cosh V(l_0-l')}{\sinh Vl_0}, \\
I'\Big]_{Z_0=\infty} &= I(\cosh Vl' - \coth Vl_0 \sinh Vl') = I\frac{\sinh V(l_0-l')}{\sinh Vl_0}.
\end{aligned}\right\} \quad (11)
$$

$$
\left.\begin{aligned}
E_0 &= \frac{I}{Y_0 \cosh Vl_0 + \frac{1}{U}\sinh Vl_0}, \\
I_0 &= \frac{I}{\cosh Vl_0 + \frac{Z_0}{U}\sinh Vl_0}.
\end{aligned}\right\} \quad (12)
$$

$$
\left.\begin{aligned}
E_0\Big]_{Z_0=\infty} &= IU \operatorname{cosech} Vl_0, \\
I_0\Big]_{Z_0=0} &= I \operatorname{sech} Vl_0.
\end{aligned}\right\} \quad (13)
$$

These equations will serve to determine the voltage at any point in a line, under any condition, and will also furnish the basis for a discussion of the phenomena of quarter-wave transmisson.

Consider first the distortion of wave shape of voltage in an open-circuited line when the generator voltage contains higher harmonics of small magnitude in comparison with the fundamental. The wave shape of the generator voltage then differs little from a sine curve. Of course, if no higher harmonics are present in the original wave shape of the generator, there will be no distortion. Referring to equation (6), suppose the line length to be one-seventh of a quarter wave length for the fundamental, or approximately, $\beta_1{}^\circ l_0 = 13$ degrees. When the line losses are small, the voltage of fundamental frequency does not change very much throughout the entire line length. The third and fifth harmonics of voltage are considerably magnified, while for the seventh harmonic, tanh Vl_0 becomes very large, infinity if α be zero, so that this particular harmonic attains values relatively much greater than either the third or the fifth. It will thus greatly alter the shape of the e.m.f. wave at points along the line, as well as at the load end. A numerical example serves best to illustrate this effect.

110. Numerical Illustration. 100 Mile, Open-circuited Line.—The 100 mile transmission line whose fundamental constants are

$r = 0.275$, $L = 0.00204$, $g = 0.15 \times 10^{-6}$, and $C = 0.0146 \times 10^{-6}$,

has, for the frequencies which are the odd multiples of 60 cycles per second, and which are the only ones which would generally be present in the e.m.f. wave of an alternator, the propagation-constants and open- and short-circuit impedances given in Tables XIX and XXI. In this case $l_0 = 100$ miles.

The coefficients of E in equations (6) may be termed the voltage and current coefficients, respectively. Numerical values of these coefficients for different frequencies and different distances l', from the generator are given below in Table XXIV.

TABLE XXIV

FACTORS FOR OBTAINING THE DISTORTION OF VOLTAGE AND CURRENT WAVE SHAPES ALONG A 100 MILE LINE WHEN OPEN-CIRCUITED

(a) Voltage coefficients, $\frac{E'}{E}$, as scalar with angle

f	$l' = 25$	$l' = 50$	$l' = 75$	$l' = 100$
60	1.009 − 0°.21	1.016 − 0°.36	1.020 − 0°.44	1.021 − 0°.47
180	1.097 − 0 .76	1.168 − 1 .25	1.211 − 1 .52	1.226 − 1 .61
300	1.387 − 2 .11	1.684 − 3. 12	1.871 − 3 .61	1.934 − 3 .76
420	3.476 − 13 .63	5.539 − 15 .81	6.895 − 16 .60	7.367 − 16 .81
540	0.654 −163 .03	2.145 −170 .69	3.192 −171 .92	3.567 −172 .20
660	0.203 − 10 .38	0.665 −174 .86	1.319 −176 .91	1.563 −177 .27
780	0.473 − 2 .51	0.260 −174 .10	0.878 −178 .41	1.119 −178 .86
900	0.678 − 1 .72	0.340 −144 .28	0.717 −179 .32	1.001 −179 .87

(b) Current coefficients, $\frac{I'}{E} \times 10^3$, as scalar with angle

f	$l' = 0$	$l' = 25$	$l' = 50$	$l' = 75$
60	0.558 88°.12	0.420 88°.05	0.281 88°.00	0.140 87°.98
180	1.898 88 .35	1.464 88 .14	0.996 87 .99	0.504 87 .90
300	4.434 86 .77	3.608 86 .38	2.546 86 .12	1.316 85 .98
420	19.545 74 .24	17.385 73 .63	12.998 73 .25	6.944 73 .04
540	9.173 −80 .50	9.387 −81 .46	7.627 −82 .00	4.262 −82 .29
660	3.222 −84 .53	4.149 −86 .19	3.785 −86 .94	2.242 −87 .30
780	1.351 −83 .64	2.714 −87 .34	2.912 −88 .40	1.855 −88 .85
900	0.182 −53 .68	1.971 −87 .68	2.676 −89 .28	1.867 −89 .83

Suppose that the generator supplying this line has an e.m.f. wave shape whose equation is, as a function of $\theta = 2\pi ft$,

$$
\begin{aligned}
e = 100{,}000 \sin \theta & \\
&+ 8000 \sin (3\theta - 40°) \\
&+ 6000 \sin (5\theta - 295°) \\
&+ 5000 \sin (7\theta - 110°) \\
&+ 2000 \sin (9\theta - 50°) \\
&+ 500 \sin (11\theta - 170°).
\end{aligned} \tag{14}
$$

The equation for any particular component of the e.m.f. at any point in the line is obtained by multiplying the equation for the component of the same frequency in e, by the factor, or voltage coefficient in part (a) of Table XXIV, and then changing the phase angle as given in (14) by the angle of the factor. Thus, at a distance of 75 miles from the generator, the maximum value of the ninth harmonic in the voltage wave is 3.192×2000 volts = 6384 volts, and its equation as a function of time (or θ) is

$$e_9\Big]_{l'=75} = 6384 \sin (9\,\theta - 221°.92).$$

111. Equations for Instantaneous Voltages. — The following are the equations for the voltages at distances of 25, 50, 75, and 100 miles from the generator.

25 miles from the generator, $l' = 25$,

$$\begin{aligned} e' = {} & 100{,}900 \sin (\theta - 0°.21) \\ & +8776 \sin (3\,\theta - 40°.76) \\ & +8322 \sin (5\,\theta - 297°.11) \\ & +17{,}380 \sin (7\,\theta - 123°.63) \\ & +1308 \sin (9\,\theta - 213°.03) \\ & +102 \sin (11\,\theta - 180°.38). \end{aligned} \tag{15}$$

50 miles from the generator, $l' = 50$,

$$\begin{aligned} e' = {} & 101{,}600 \sin (\theta - 0°.36) \\ & +9344 \sin (3\,\theta - 41°.25) \\ & +10{,}104 \sin (5\,\theta - 298°.12) \\ & +27{,}700 \sin (7\,\theta - 125°.81) \\ & +4290 \sin (9\,\theta - 220°.69) \\ & +332 \sin (11\,\theta - 344°.86). \end{aligned} \tag{16}$$

75 miles from the generator, $l' = 75$,

$$\begin{aligned} e' = {} & 102{,}000 \sin (\theta - 0°.44) \\ & +9688 \sin (3\,\theta - 41°.52) \\ & +11{,}226 \sin (5\,\theta - 298°.61) \\ & +34{,}480 \sin (7\,\theta - 126.°60) \\ & +6384 \sin (9\,\theta - 221.°92) \\ & +660 \sin (11\,\theta - 346°.91). \end{aligned} \tag{17}$$

100 miles from the generator, $l' = 100$, at open end,

$$\begin{aligned} e' = 102{,}100 \sin (\theta - 0^\circ.47) \\ +9808 \sin (3\,\theta - 41^\circ.61) \\ +11{,}604 \sin (5\,\theta - 298^\circ.76) \\ +36{,}835 \sin (7\,\theta - 126^\circ.81) \\ +7134 \sin (9\,\theta - 222^\circ.20) \\ +782 \sin (11\,\theta - 347^\circ.27). \end{aligned} \tag{18}$$

112. Equations for Instantaneous Currents. — Using the current coefficients from part (*b*), Table XXIV, the following equations are obtained for the line currents at the several different distances from the generator. Obviously, the current is zero at the open end of the line, where $l' = l_0 = 100$.

Generator current, at $l' = 0$,

$$\begin{aligned} i = 55{,}800 \sin (\theta + 88^\circ.12) \\ +15{,}184 \sin (3\,\theta + 48^\circ.35) \\ +26{,}604 \sin (5\,\theta - 208^\circ.23) \\ +97{,}725 \sin (7\,\theta - 35^\circ.76) \\ +18{,}346 \sin (9\,\theta - 130^\circ.50) \\ +1611 \sin (11\,\theta - 254^\circ.53). \end{aligned} \tag{19}$$

25 miles from the generator, $l' = 25$,

$$\begin{aligned} i' = 42{,}000 \sin (\theta + 88^\circ.05) \\ +11{,}712 \sin (3\,\theta + 48^\circ.14) \\ +21{,}648 \sin (5\,\theta - 208^\circ.62) \\ +86{,}925 \sin (7\,\theta - 36^\circ.37) \\ +18{,}774 \sin (9\,\theta - 131^\circ.46) \\ +2074 \sin (11\,\theta - 256^\circ.19). \end{aligned} \tag{20}$$

50 miles from the generator, $l' = 50$,

$$\begin{aligned} i' = 28{,}100 \sin (\theta + 88^\circ.00) \\ +7968 \sin (3\,\theta + 47^\circ.99) \\ +15{,}276 \sin (5\,\theta - 208^\circ.88) \\ +64{,}990 \sin (7\,\theta - 36^\circ.75) \\ +15{,}254 \sin (9\,\theta - 132^\circ.00) \\ +1892 \sin (11\,\theta - 256^\circ.94). \end{aligned} \tag{21}$$

75 miles from the generator, $l' = 75$,

$$\begin{aligned} i' = 14{,}060 \sin (\theta + 87^\circ.98) \\ +4032 \sin (3\,\theta + 47^\circ.90) \\ +7896 \sin (5\,\theta - 209.^\circ02) \\ +34{,}720 \sin (7\,\theta - 36^\circ.96) \\ +8524 \sin (9\,\theta - 132^\circ.29) \\ +1121 \sin (11\,\theta - 257^\circ.30). \end{aligned} \tag{22}$$

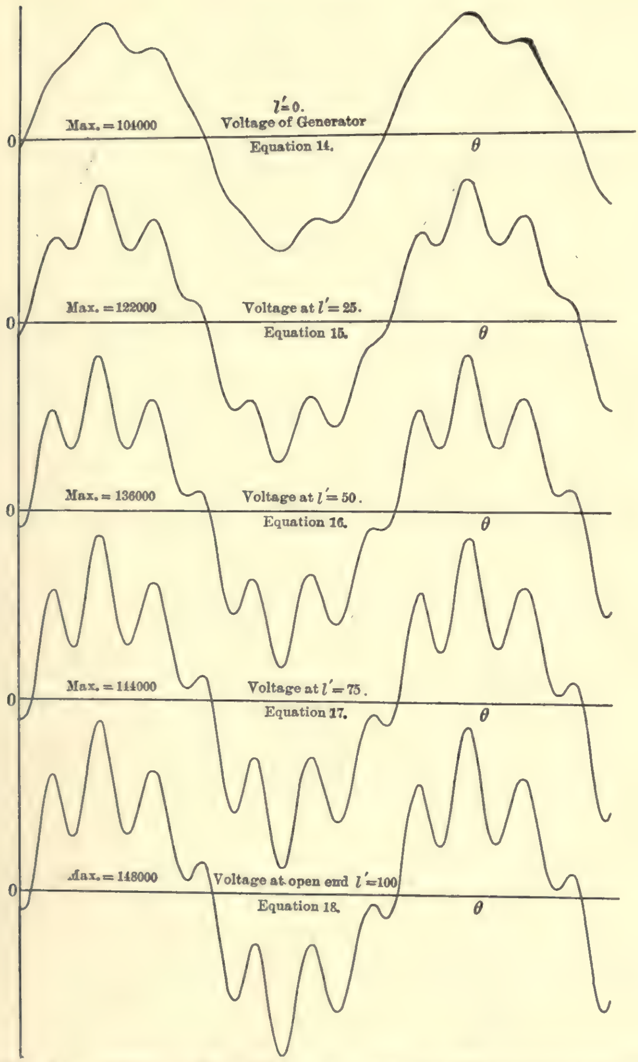

FIG. 64. — Voltage Wave Shapes along a 100 Mile Open-circuited Power Line.

113. Discussions of Wave Shapes. — The curves representing the foregoing equations are shown in Figs. 64 and 64a. In both figures, the upper curve represents the generator voltage wave shape. These curves were drawn by means of the curve-tracer

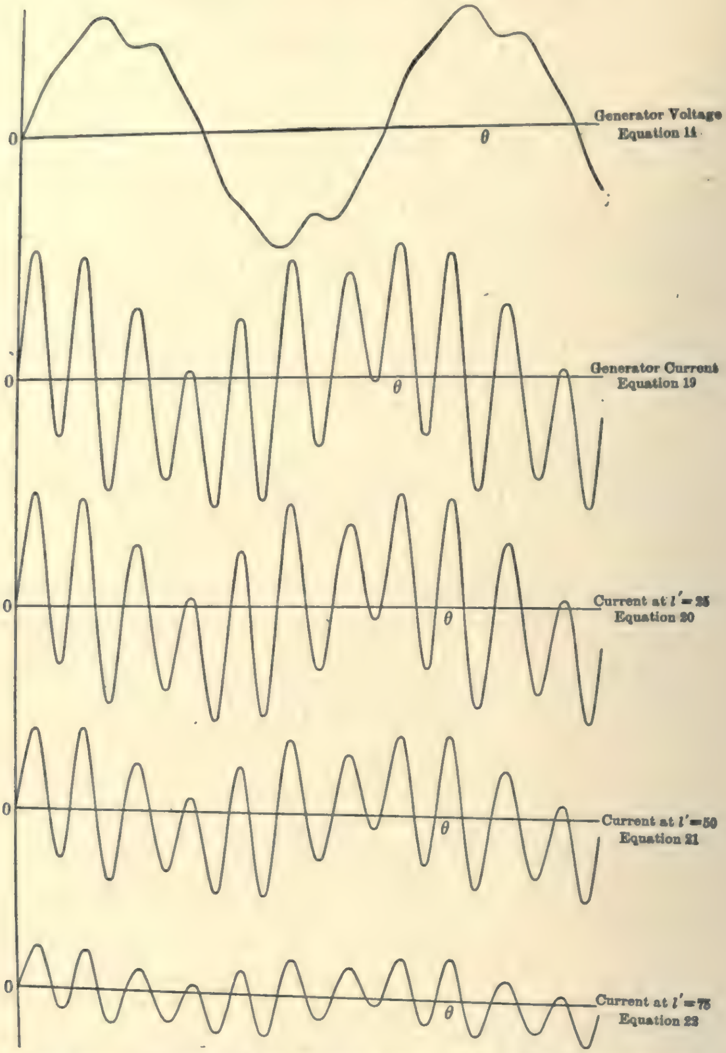

FIG. 64a. — Current Wave Shapes along a 100 Mile Open-circuited Power Line.

mentioned in section 42, Chapter III, and, though they may be slightly inaccurate, serve sufficiently well to show the characteristics of the different wave shapes.

In the particular line to which the foregoing curves apply, the

total length of circuit is 100 miles, and therefore approximately equal to one-quarter of the wave length for the seventh harmonic — 109 miles, from the value of $\beta°$ given in Table XIX. The voltage and current coefficients for this frequency attain values much larger than for any of the other frequencies, and therefore the comparatively small seventh harmonic in the generator voltage produces very prominent voltages and currents of this frequency throughout the line. Although the seventh harmonic in the generator voltage is only five per cent of the fundamental, the seventh harmonic in the voltage at the free end of the line is thirty-six per cent of the fundamental. In the capacity, or charging, current of the line, the seventh harmonic component is approximately twice as great as the fundamental.

The maximum instantaneous values of voltage at different points along the line are indicated on the respective curves. On account of the different phase positions of the several component harmonics, the maximum value of the generator voltage wave is only a little greater than the maximum value of the fundamental — 104,000 maximum, with 100,000 volts as maximum value of the fundamental. At the open end of the line, the maximum value of the voltage wave is 148,000 volts, which, as an increase of nearly fifty per cent over the generator voltage, might be serious in its effect upon the line insulation. In a line designed to operate normally without corona formation at the voltage assigned to the fundamental frequency in this illustrative example, the presence of such a large superimposed harmonic which increases the maximum voltage by nearly fifty per cent might be sufficient to start corona formation, and introduce a power loss then supplied, in part, by the fundamental frequency. Very considerable values of leakage, or corona loss, are required to materially decrease the voltage coefficients and thus to decrease the exaggeration of those harmonics for which the line approximates a quarter wave length. The value of g used in the computation of the data for this particular case is 0.15×10^{-6}, which corresponds to a power loss due to leakage, of 1500 watts per mile of each wire for an e.m.f. of 100,000 volts between wire and neutral. For a line differing, as in this case, by 9 miles from a quarter wave length for the seventh harmonic, this value of g is not sufficient to appreciably lower the value of the voltage coefficient in Table XXIV.

114. Dependence of Wave Shapes upon Generator and Load Characteristics. — The large value of the seventh harmonic in the generator current would seriously increase the generator heating, and demands, even under conditions of no-load, considerable generator capacity. The seventh harmonic current at the generator is 74.2 degrees ahead of the corresponding component in the terminal voltage, so that, for a considerable range of current values, whatever current of this frequency flowed through the inductive impedance of the generator windings would produce an increase of the terminal voltage (of this frequency) above the open-circuit value. No reliance could be placed on an assumption that in such a case the generator terminal voltage of this frequency would decrease greatly when the current flows. For a line a little longer than a quarter wave length, the current would lag by nearly 90 degrees, though the numerical value of the voltage coefficient might be the same as before. In such a case, the terminal voltage of the generator undoubtedly would decrease materially as soon as the current flowed.

An increase of only nine per cent in the frequency (from 60 to 65 cycles for the fundamental) would cause the line to be an exact quarter wave length for the seventh harmonic, in which event, much greater distortion of the wave shapes would occur. A still further increase of the frequency would cause the line length to be greater than a quarter wave length for the seventh harmonic, and, depending upon the character of the source of e.m.f. of this frequency, might entirely alter the phenomena.

The presence of a load at the terminal of such a line would also change the phenomena. If the load consisted of an inert impedance, Z_0, the distribution of electrical quantities along the line could be determined by equations (4), although not without rather tedious computations. If the load consists of a synchronous motor, for example, instead of an inert impedance, it is necessary to know the wave shape of counter electromotive force of this machine, and furthermore, to know the relative phase positions of its component harmonics of e.m.f. with respect to those of the generator supplying the line. The counter e.m.f. of fundamental frequency of the motor will be nearly in opposition to the e.m.f. impressed on the motor terminals, E_0, but, by reference to Fig. 33 it is seen that the phase angle between the generator terminal voltage, E, and the load voltage, E_0, may differ by large amounts,

depending upon the magnitude of the current I_0 as well as upon the load power-factor. The possibility of a variation of, say, 60 degrees in the phase difference between E and E_0, allows the higher harmonics in the counter e.m.f. of the motor to occupy almost any phase position whatsoever with respect to the same harmonic frequencies in the generator e.m.f.

If a particular harmonic is present in the generator wave shape, but is not present in the counter e.m.f. of the motor, then the motor constitutes practically a short-circuit at the load end of the line for this particular frequency. Similarly, the generator constitutes a short-circuit at its end for any harmonic present in the motor counter e.m.f. which is not present in the generator wave. The discussion of the mode of operation of the various machines which may be connected to a line must be taken up in another place; this section is intended to deal only with the calculation of the values of the voltage and current along the line when the terminal conditions are given.

If both the generator and the load produce e.m.f.'s, as in the case of a higher harmonic in both the generator and motor terminal e.m.f.'s, the distribution of voltage and current along the line must be determined by equations identical with (55) and (56), or (57) and (58), of Chapter II; in which, to conform to the notation used here for alternating current systems, the complex

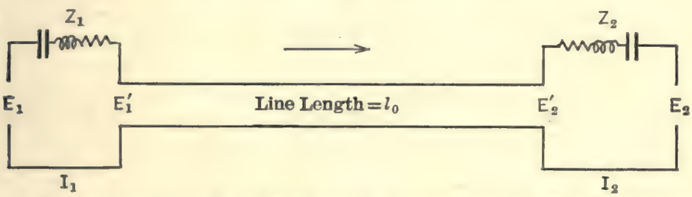

FIG. 65. — Long Line Supplied through Impedances by Two Generators.

quantities, U and V, must be substituted for z and v, respectively. E_1 and E_2 are then the values (in the correct complex form to take account of their relative phase positions) of the voltages impressed on the two ends of the line, and E_0 and I' and I'' are respectively the values of voltage and current at a distance l_1 from E_1 and l_2 from E_2.

115. Double Source of Energy Supply Through Terminal Impedances. — If impedances are placed between the sources of e.m.f. and the line, the circuit appears as in Fig. 65, and the solu-

tion must be obtained as below. In this system, select arbitrarily as positive the direction indicated by the arrow. Then, by Kirchhoff's Law,

$$\left.\begin{aligned} E_1' &= E_1 - I_1Z_1, \\ E_2' &= I_2Z_2 - E_2. \end{aligned}\right\} \tag{23}$$

By equations (2),

$$\left.\begin{aligned} E_2' &= E_1' \cosh Vl_0 - UI_1 \sinh Vl_0, \\ I_2 &= I_1 \cosh Vl_0 - \frac{1}{U} E_1' \sinh Vl_0, \end{aligned}\right\} \tag{24}$$

or, substituting (23) in (24) and combining terms,

$$\left.\begin{aligned} I_1 [Z_1 \cosh Vl_0 + U \sinh Vl_0] + Z_2I_2 &= E_1 \cosh Vl_0 + E_2, \\ I_1\left[\cosh Vl_0 + \frac{Z_1}{U} \sinh Vl_0\right] - I_2 &= E_1 \frac{1}{U} \sinh Vl_0, \end{aligned}\right\} \tag{25}$$

from which, by solving simultaneously for I_1 and I_2,

$$\left.\begin{aligned} I_1 &= \frac{E_1\left(\cosh Vl_0 + \frac{Z_2}{U} \sinh Vl_0\right) + E_2}{(Z_1 + Z_2) \cosh Vl_0 + \left(U + \frac{Z_1Z_2}{U}\right) \sinh Vl_0}, \\ I_2 &= \frac{E_1 + E_2\left(\cosh Vl_0 + \frac{Z_1}{U} \sinh Vl_0\right)}{(Z_1 + Z_2) \cosh Vl_0 + \left(U + \frac{Z_1Z_2}{U}\right) \sinh Vl_0}. \end{aligned}\right\} \tag{26}$$

Using the value of I_1 from equations (26), the voltage and current at any point in the line are then given by

$$\left.\begin{aligned} E &= E_1' \cosh Vl - I_1U \sinh Vl, \\ I &= I_1 \cosh Vl - E_1' \frac{1}{U} \sinh Vl, \end{aligned}\right\} \tag{27}$$

where l is the distance of the point in question from the e.m.f. E_1, and $E_1' = E_1 - Z_1I_1$.

It is only rarely that conditions in any transmission system would be definitely enough known to warrant the use of equations (26) and (27). Since all of the quantities entering these equations are complex, actual numerical computations from them are quite tedious, though not difficult.

116. Approximations Based on the Neglect of Line Losses. — In many cases it is not as important to know the actual values of the different harmonic voltages or currents along the line, as it is to be able to determine quickly the approximate maximum values which these quantities may attain. The presence of resistance and leakage in the line, leading to energy loss, serves always to prevent the attainment of infinite values of current and voltage, and, in general, serves to decrease the magnification of any particular harmonic, due to resonance conditions. Therefore, calculations made for a line with zero resistance and leakage, but with the same values of L and C as in an actual system, will yield values for the current and voltage distortion greater than could ever take place in the actual system, and thus serve as a guide in determining whether or not it would be necessary to make a rigid solution.

Neglecting, then, the losses,

$$V = j\beta = j\,2\pi f\sqrt{LC}, \quad \beta = 2\pi f\sqrt{LC}, \quad \text{and} \quad U = \sqrt{\frac{L}{C}},$$

so that, by $\cosh j\beta l = \cos \beta l$, and $\sinh j\beta l = j \sin \beta l$, equations (5) to (13) become,

$$\left.\begin{aligned} E'\Big]_{Z_0=0} &= E\frac{\sin \beta\,(l_0 - l')}{\sin \beta l_0}, \\ I'\Big]_{Z_0=0} &= -jE\sqrt{\frac{C}{L}}\frac{\cos \beta\,(l_0 - l')}{\sin \beta l_0}. \end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned} E'\Big]_{Z_0=\infty} &= E\frac{\cos \beta\,(l_0 - l')}{\cos \beta l_0}, \\ I'\Big]_{Z_0=\infty} &= jE\sqrt{\frac{C}{L}}\frac{\sin \beta\,(l_0 - l')}{\cos \beta l_0}. \end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned} E_0 &= \frac{E}{\cos \beta l_0 + j\sqrt{\frac{L}{C}}\,Y_0 \sin \beta l_0}, \\ I_0 &= \frac{E}{Z_0 \cos \beta l_0 + j\sqrt{\frac{L}{C}} \sin \beta l_0}. \end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned} E'\Big]_{Z_0=0} &= jI\sqrt{\frac{L}{C}}\frac{\sin\beta\,(l_0 - l')}{\cos\beta l_0},\\ I'\Big]_{Z_0=0} &= I\frac{\cos\beta\,(l_0 - l')}{\cos\beta l_0}. \end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned} E'\Big]_{Z_0=\infty} &= -jI\sqrt{\frac{L}{C}}\frac{\cos\beta\,(l_0 - l')}{\sin\beta l_0},\\ I'\Big]_{Z_0=\infty} &= I\,\frac{\sin\beta\,(l_0 - l')}{\sin\beta l_0}. \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} E_0 &= \frac{I}{Y_0\cos\beta l_0 + j\sqrt{\frac{C}{L}}\sin\beta l_0},\\ I_0 &= \frac{I}{\cos\beta l_0 + jZ_0\sqrt{\frac{C}{L}}\sin\beta l_0}. \end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} E_0\Big]_{Z_0=\infty} &= E\sec\beta l_0,\\ I_0\Big]_{Z_0=0} &= -jE\sqrt{\frac{C}{L}}\operatorname{cosec}\beta l_0. \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} E_0\Big]_{Z_0=\infty} &= -jI\sqrt{\frac{L}{C}}\operatorname{cosec}\beta l_0,\\ I_0\Big]_{Z_0=0} &= I\sec\beta l_0. \end{aligned}\right\} \tag{35}$$

For either short-circuited or open-circuited lines, computation by the foregoing approximate equations is very easy and rapid. It is not proper, of course, to use them when they reduce to either infinite or zero values, for then the line resistance and leakage become the predominating quantities which control the distribution of line voltage and current throughout. In the simple cases of short-circuited and open-circuited lines, the equations may be used except for values of βl approximating multiples of 90 degrees. In this event, since for power lines α is approximately constant for frequencies above 60 cycles per second, the magnitudes of the phenomena at these critical frequencies may be easily determined from equations (5) to (13) by the simplified expressions,

$$\left.\begin{aligned}
\cosh\left(\alpha l + j\,\frac{\pi}{2}\right) &= j\sinh \alpha l.\\
\cosh\left(\alpha l + j\,\pi\right) &= -\cosh \alpha l.\\
\cosh\left(\alpha l + j\,\frac{3\,\pi}{2}\right) &= -j\sinh \alpha l.\\[1ex]
\sinh\left(\alpha l + j\,\frac{\pi}{2}\right) &= j\cosh \alpha l.\\
\sinh\left(\alpha l + j\,\pi\right) &= -\sinh \alpha l.\\
\sinh\left(\alpha l + j\,\frac{3\,\pi}{2}\right) &= -j\cosh \alpha l.\\[1ex]
\tanh\left(\alpha l + j\,\frac{\pi}{2}\right) &= \coth \alpha l.\\
\tanh\left(\alpha l + j\,\pi\right) &= \tanh \alpha l.\\
\tanh\left(\alpha l + j\,\frac{3\,\pi}{2}\right) &= \coth \alpha l.
\end{aligned}\right\} \qquad (36)$$

The above formulæ are directly applicable to the discussion of quarter-wave transmission, where the frequency of the generator is adjusted to such a value that the line becomes a quarter wave-length.

117. Numerical Comparison of Approximate and Exact Solutions. — Applying the approximations, equations (29), to the line whose voltage and current coefficients for the different frequencies are given in Table XXIV, the following numerical results are obtained, which, as approximations, may be compared with the accurate values given in Table XXIV. In computing the following, the value of $2\,\pi f\sqrt{LC}$ is used for β, and not the true values, dependent upon the line resistance and leakage as given in Table XIX.

In the following table, the complex quantity representing the designated ratios is given in the form of a scalar and its angle.

In all cases, except for a frequency of 420 cycles per second, for which the line is nearly a quarter wave length, the approximations are sufficiently close to the true results for all, or at least most, engineering purposes, and the labor required to secure the approximate results is practically negligible in comparison with that required by the rigid expressions.

For this line, by equations (13) and (36) when the frequency is slightly higher than 420 cycles per second, so that βl_0 is exactly $\frac{\pi}{2}$,

the voltage at the open-circuited load end is 25.26 times the generator voltage.

TABLE XXV

APPROXIMATE VALUES OF VOLTAGE AND CURRENT COEFFICIENTS FROM EQUATION (29) FOR A DISTANCE (l') OF 75 MILES FROM THE GENERATOR.

Total line length = 100 Miles. True values of coefficients taken from Table XXIV for comparison.

f	$\frac{E'}{E}$, true		$\frac{E'}{E}$, approx.		$\frac{I'}{E}10^3$, true		$\frac{I'}{E}10^3$, approx.	
60	1.020	$-\ 0°.44$	1.020	0°	0.140	87°.98	0.141	90°
180	1.211	$-\ 1\ .52$	1.212	0	0.504	87 .90	0.504	90
300	1.871	$-\ 3\ .61$	1.874	0	1.316	85 .98	1.319	90
420	6.895	$-\ 16\ .60$	9.627	0	6.944	73 .04	9.943	90
540	3.192	$-171\ .92$	3.228	180	4.262	$-82\ .29$	4.310	-90
660	1.319	$-176\ .91$	1.322	180	2.242	$-87\ .30$	2.247	-90
780	0.878	$-178\ .41$	0.879	180	1.855	$-88\ .85$	1.857	-90
900	0.717	$-179\ .32$	0.718	180	1.867	$-89\ .83$	1.868	-90

118. Resonance Frequencies.—In the general case, with the line closed through impedances at both ends, the frequencies at which resonance phenomena occur may be easily determined from equation (26). When resonance phenomena occur, the line currents become very large, even for small impressed electromotive forces, and in the limiting case, when neither resistance nor leakage is present, and consequently no losses, the currents become infinite with finite voltages impressed. The only way in which these infinite values of I_1 and I_2 can occur is for the denominator in equation (26) to become zero. That is, the values of f which cause the denominator of this equation to reduce to zero, when the line losses are assumed zero, are those at which the resonance phenomena occur. Thus we have for resonance, from (26),

$$(Z_1 + Z_2)\cos\beta l_0 + j\left(\sqrt{\frac{L}{C}} + \sqrt{\frac{C}{L}}\,Z_1 Z_2\right)\sin\beta l_0 = 0. \qquad (37)$$

Z_1 and Z_2 are functions of the frequency, and though they may be made up of any combinations of inductances and capacities, no conductance or resistance should be present if the above equation is to apply rigorously. β also is a function of the frequency, $\beta = 2\pi f\sqrt{LC}$. The solution of equation (37) for f must be

obtained by approximation — a graphical process similar to that used in determining the free frequencies of vibration for the oscillograph vibrator, Chapter V, Fig. 21, being convenient.

To illustrate the use of this equation in determining the critical, or resonance, frequencies of a line connected as indicated in Fig. 65, consider again the same 100 mile power transmission circuit. Let one end be closed through an inductance of 0.050 henry, and the other through a condenser of 0.020 microfarad and an inductance of 0.10 henry in series.

Then

$$\left.\begin{aligned} Z_1 &= j\,2\pi f \times 0.050, \\ Z_2 &= j\left(2\pi f \times 0.10 - \frac{10^6}{2\pi f \times 0.02}\right). \end{aligned}\right\} \tag{38}$$

For the line, per mile,

$$L = 0.00204 \qquad \text{and} \qquad C = 0.0146 \times 10^{-6}.$$

Substituting these numerical values in (37) and expressing the result in such form as to utilize the tangent, we have, finally,

$$\tan 0^\circ.031269\,\omega = \frac{1}{\omega}\,\frac{50 \times 10^6 - 0.15\,\omega^2}{7061.8 - 0.000018376\,\omega^2}. \tag{39}$$

If we let y' represent the first and y'' the second member of this equation, the intersections of the curves y' and y'' plotted as functions of ω will occur at those values of ω which satisfy the above equation and which therefore are those corresponding to resonance conditions, or frequencies. These curves are shown in Fig. 66, from which intersections at the following values of ω are noted:

ω, for resonance	$\omega/2\pi$ = frequency
2,302	366
7,200	1150
12,400	1970
17,600	2800
19,700	3130
23,800	3780
29,300	4660
33,000	5250
....	....
....	....

Only the first two of these frequencies are sufficiently low to fall within the range of the higher harmonics produced by alternating current generators. It is interesting to note that the first resonance frequency, 366 cycles per second, is considerably lower than that for the open-circuited line alone, when βl_0 becomes $\frac{\pi}{2}$, or 90 degrees, at a frequency of 458 cycles per second.

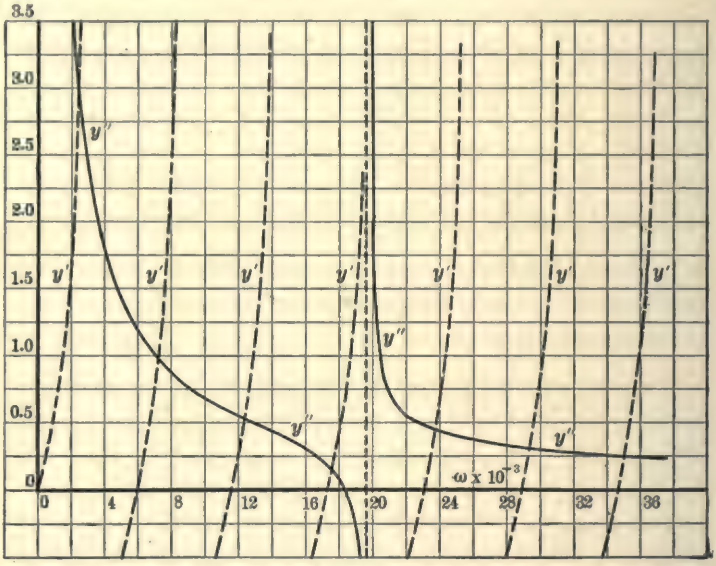

FIG. 66. — Graphical Determination of Resonance Frequencies from Equation (39).

Even with this comparatively short line — 100 miles — and the small value of electrostatic capacity placed at one end of the circuit, this first resonance frequency of 366 cycles per second is only the sixth harmonic in a 60 cycle system. An increase in the capacity and self-inductance present would cause resonance at a still lower frequency, and since the fifth harmonic is quite prevalent in generator voltage wave shapes, resonance phenomena of this frequency may well appear in a one hundred mile transmission system. For line lengths between 200 and 300 miles, the first resonance frequency appears at about the third harmonic of 60 cycles, or 180 cycles per second, so that resonance phenomena of this frequency may be produced if any source of e.m.f. of this frequency is present.

From equation (37), used in determining the resonance frequencies in this particular case, it is apparent that it makes no difference whether the terminal impedances are inserted with Z_1 at the generator end and Z_2 at the load end, or vice versa. Resonance phenomena appear at the same frequency in both cases, since Z_1 and Z_2 enter equation (37) only as sums or products.

Even if the magnitudes of voltage and current produced by resonance condition for one of these higher harmonics should not be such as to endanger the power system, or even to be observed in their effect on normal operating conditions, they still may be of sufficient magnitude to create serious disturbance in communicating circuits which parallel the power system. For this reason, particularly, it becomes desirable, or necessary in some cases, to eliminate, as far as may be, the possibility of abnormal values of voltage or current of these higher frequencies in power systems. The large magnitudes of voltage and current of fundamental frequency are not so serious in their effect upon communication circuits as the smaller voltages and currents of the higher frequencies.

CHAPTER X

POWER RELATIONS IN LONG LINES

119. Instantaneous and Average Power. — The instantaneous value of power at any point in a transmission line is equal to the product of the instantaneous values of voltage and current. In Fig. 31 and section 70 of Chapter VI is given an illustration of the instantaneous voltage, current, and power distribution along a line. The average value of power throughout a cycle is of more interest and importance.

Explicit expressions (equations (3), (8), (18), (19), etc., Chapter VI) have been given for the voltage and current at any point in a line, and from these the equation for average power may be derived by taking the product of the scalar values of these quantities times the cosine of the angle of phase difference between them,

$$P = ei \cos \phi.$$

By the relation that the cosine of the angle between two vectors is equal to the sum of the products of the direction-cosines of the two vectors, referred to two axes at right angles, a more convenient form of expression may be secured. Thus, referring to the e.m.f. and current represented analytically by

$$E = e_1 + je_2 \qquad \text{and} \qquad I = i_1 + ji_2,$$

we have

$$P = \text{average power} = ei \cos \phi = e_1 i_1 + e_2 i_2. \tag{1}$$

120. The Separate Components of E and I at any Point. — Obviously, then, the problem of setting up an explicit expression for the average power at any point in a line resolves itself into obtaining expressions for the separate components of E and I at any point and then forming the sum of the products of in-phase components. The details of such a development are quite simple, but nevertheless tedious.

The quantities entering into the expressions for voltage and current are

$$\left.\begin{aligned} V &= \alpha + j\beta, \\ U &= \sqrt{\frac{Z}{Y}} = u\underline{/\theta_u} = u_1 + ju_2, \\ \frac{1}{U} &= \sqrt{\frac{Y}{Z}} = \frac{u_1}{u^2} - j\frac{u_2}{u^2} = y_1 + jy_2. \end{aligned}\right\} \qquad (2)$$

At the load end, let

$$E = e_0' + je_0'' \quad \text{and} \quad I = i_0' + ji_0''.$$

Then by equation (19), Chapter VI, the voltage and current at any point in the line distant l from the load end are given by

$$\left.\begin{aligned} E = {} & (e_0' + je_0'')(\cosh \alpha l \cos \beta l + j \sinh \alpha l \sin \beta l) \\ & + (i_0' + ji_0'')(u_1 + ju_2)(\sinh \alpha l \cos \beta l + j \cosh \alpha l \sin \beta l), \\ I = {} & (i_0' + ji_0'')(\cosh \alpha l \cos \beta l + j \sinh \alpha l \sin \beta l) \\ & + (e_0' + je_0'')(y_1 + jy_2)(\sinh \alpha l \cos \beta l + j \cosh \alpha l \sin \beta l). \end{aligned}\right\} \qquad (3)$$

These quantities are of the form

$$E = e_1 + je_2 \quad \text{and} \quad I = i_1 + ji_2,$$

and by multiplying in equation (3), we have

$$\left.\begin{aligned} e_1 = {} & e_0' \cosh \alpha l \cos \beta l - e_0'' \sinh \alpha l \sin \beta l \\ & + (i_0'u_1 - i_0''u_2) \sinh \alpha l \cos \beta l \\ & - (i_0'u_2 + i_0''u_1) \cosh \alpha l \sin \beta l, \\ e_2 = {} & e_0'' \cosh \alpha l \cos \beta l + e_0' \sinh \alpha l \sin \beta l \\ & + (i_0''u_1 + i_0'u_2) \sinh \alpha l \cos \beta l \\ & + (i_0'u_1 - i_0''u_2) \cosh \alpha l \sin \beta l. \end{aligned}\right\} \qquad (4)$$

$$\left.\begin{aligned} i_1 = {} & i_0' \cosh \alpha l \cos \beta l - i_0'' \sinh \alpha l \sin \beta l \\ & + (e_0'y_1 - e_0''y_2) \sinh \alpha l \cos \beta l \\ & - (e_0'y_2 + e_0''y_1) \cosh \alpha l \sin \beta l, \\ i_2 = {} & i_0'' \cosh \alpha l \cos \beta l + i_0' \sinh \alpha l \sin \beta l \\ & + (e_0''y_1 + e_0'y_2) \sinh \alpha l \cos \beta l \\ & + (e_0'y_1 - e_0''y_2) \cosh \alpha l \sin \beta l. \end{aligned}\right\} \qquad (5)$$

121. The Power Equation. — Substituting the above expressions for the separate components in the equation

$$P = e_1 i_1 + e_2 i_2, \qquad (6)$$

multiplying out, and finally combining terms, there results the following form:

$$\begin{aligned} P = &(e_0'i_0' + e_0''i_0'')(\cosh^2 \alpha l \cos^2 \beta l + \sinh^2 \alpha l \sin^2 \beta l) \\ &+ [(e_0'^2 + e_0''^2)\, y_1 + (i_0'^2 + i_0''^2)\, u_1] \cosh \alpha l \sinh \alpha l \\ &- [(e_0'^2 + e_0''^2)\, y_2 + (i_0'^2 + i_0''^2)\, u_2] \cos \beta l \sin \beta l \\ &+ [(e_0'i_0' + e_0''i_0'')(u_1 y_1 + u_2 y_2) + (e_0'i_0'' - e_0''i_0') \\ &\quad (u_1 y_2 - u_2 y_1)](\cosh^2 \alpha l \sin^2 \beta l + \sinh^2 \alpha l \cos^2 \beta l). \qquad (7) \end{aligned}$$

Considerable care is required in reducing the product as indicated by equation (6) to the form given in (7) on account of the large number of terms involved.

The power at the load end, or receiver, is

$$P_0 = e_0'i_0' + e_0''i_0''$$

and the scalar values of voltage and current at the load end are

$$e_0^2 = e_0'^2 + e_0''^2 \quad \text{and} \quad i_0^2 = i_0'^2 + i_0''^2.$$

Introducing these values into equation (7),

$$\begin{aligned} P = &P_0 (\cosh^2 \alpha l \cos^2 \beta l + \sinh^2 \alpha l \sin^2 \beta l) \\ &+ (e_0^2 y_1 + i_0^2 u_1) \cosh \alpha l \sinh \alpha l \\ &- (e_0^2 y_2 + i_0^2 u_2) \cos \beta l \sin \beta l \\ &+ [P_0 (u_1 y_1 + u_2 y_2) + (e_0'i_0'' - e_0''i_0') \\ &\quad (u_1 y_2 - u_2 y_1)](\cosh^2 \alpha l \sin^2 \beta l + \sinh^2 \alpha l \cos^2 \beta l). \qquad (8) \end{aligned}$$

122. Reduction in Form of the Power Equation. — Still further reduction is possible by a consideration of the necessary relations existing between u_1 and u_2, and y_1 and y_2.

$$U = u_1 + ju_2 = \sqrt{\frac{Z}{Y}} = u\underline{/\theta_u}, \quad u^2 = u_1^2 + u_2^2.$$

$$\frac{1}{U} = y_1 + jy_2 = \frac{u_1}{u_1^2 + u_2^2} - j\frac{u_2}{u_1^2 + u_2^2}.$$

$$y_1 = \frac{u_1}{u^2} \quad \text{and} \quad y_2 = -\frac{u_2}{u^2}.$$

Thus

$$\left.\begin{aligned} &u_1 y_1 + u_2 y_2 = \frac{u_1^2 - u_2^2}{u_1^2 + u_2^2} = 1 - 2\frac{u_2^2}{u^2} = 1 + 2\, u_2 y_2 \\ &\text{and} \\ &u_1 y_2 - u_2 y_1 = -2\frac{u_1 u_2}{u^2} = +2\, u_1 y_2. \end{aligned}\right\} \qquad (9)$$

Introducing these expressions into equation (8),

$$\begin{aligned} P = P_0 &(\cosh^2 \alpha l \cos^2 \beta l + \sinh^2 \alpha l \sin^2 \beta l) \\ &+ (e_0^2 y_1 + i_0^2 u_1) \cosh \alpha l \sinh \alpha l \\ &- (e_0^2 y_2 + i_0^2 u_2) \cos \beta l \sin \beta l \\ &\{P_0 + 2 [P_0 u_2 y_2 + (e_0' i_0'' - e_0'' i_0') u_1 y_2]\} \\ &(\cosh^2 \alpha l \sin^2 \beta l + \sinh^2 \alpha l \cos^2 \beta l). \end{aligned} \tag{10}$$

Combining the first term, and the portion of the fourth term in the right hand member which contains P_0 alone, there results

$$P_0 [\cosh^2 \alpha l (\cos^2 \beta l + \sin^2 \beta l) + \sinh^2 \alpha l (\cos^2 \beta l + \sin^2 \beta l)],$$

which reduces to

$$P_0 (\cosh^2 \alpha l + \sinh^2 \alpha l) \text{ or } P_0 (1 + 2 \sinh^2 \alpha l).$$

The expression

$$e_0' i_0'' - e_0'' i_0' = P_0^j \tag{11}$$

represents the wattless volt-amperes at the load end (see Steinmetz' "Alternating Current Phenomena," page 218), and is commonly designated by the symbol, P^j. Numerically, also,

$$P_0^j = e_0 i_0 \sin \phi_0,$$

where ϕ_0 is the power-factor angle of the load, counted, in the vector diagram, from E_0 to I_0 in a counter-clockwise direction, and is thus positive in value for a leading (condensive) load. The real power is

$$P_0 = e_0 i_0 \cos \phi,$$

so that

$$\text{volt-amperes} = e_0 i_0 = \sqrt{P_0^2 + P_0^{j2}}.$$

In further writing, products of the form

$$uv \cos (\theta_v - \theta_u) = u_1 v_1 + u_2 v_2$$

may be written in the abreviated notation, $U \cdot V$; products of the form

$$uv \sin (\theta_v - \theta_u) = u_1 v_2 - u_2 v_1$$

may be written with the notation, $U \times V$.

$$\begin{aligned} U \cdot V &\equiv uv \cos (\theta_v - \theta_v) = u_1 v_1 + u_2 v_2, \\ U \times V &\equiv uv \sin (\theta_v - \theta_u) = u_1 v_2 - u_2 v_1, \end{aligned}$$

where

$$U = u_1 + ju_2, \quad V = v_1 + jv_2,$$
$$UV \equiv (u_1 + ju_2)(v_1 + jv_2),$$
$$uv = \sqrt{u_1^2 + u_2^2}\ \sqrt{v_1^2 + v_2^2}.$$

Therefore

$$(uv)^2 = (U \cdot V)^2 + (U \times V)^2.$$

For example,

$$P = E \cdot I, \quad P^j = E \times I, \quad \text{and} \quad \text{volt-amperes} = ei.$$

123. Final Form of Power Equation. — For further reduction

$$\cosh \alpha l \sinh \alpha l = \tfrac{1}{2} \sinh 2\,\alpha l,$$
$$\cos \beta l \sin \beta l \quad = \tfrac{1}{2} \sin 2\,\beta l,$$

and

$$\cosh^2 \alpha l \sin^2 \beta l + \sinh^2 \alpha l \cos^2 \beta l = \sinh^2 \alpha l + \sin^2 \beta l,$$

so that the final form may be taken to be

$$\begin{aligned} P = P_0 (1 + 2 \sinh^2 \alpha l) &+ (e_0^2 y_1 + i_0^2 u_1) \sinh \alpha l \cosh \alpha l \\ &- (e_0^2 y_2 + i_0^2 u_2) \sin \beta l \cos \beta l \\ &+ 2 (P_0 u_2 y_2 + P_0^j u_1 y_2)(\sinh^2 \alpha l + \sin^2 \beta l). \end{aligned} \quad (12)$$

124. Numerical Example. — As an illustration of this final equation, let us compute the power at the generator for a 100 mile transmission line with the constants

$$r = 0.275, \quad g = 0.15 \times 10^{-6}, \quad L = 0.00204, \quad \text{and} \quad C = 0.0146 \times 10^{-6}$$

per mile, when the e.m.f. and current at the load end are respectively 100,000 volts and 200 amperes, and the load power-factor angle is $\phi_0 = -25.0$ degrees. This signifies a "lagging load." Let the frequency be 25 cycles per second, and in order to use the equation in its general form, let neither E_0 nor I_0 be selected as reference vector, but assume

$$E_0 = e_0' + je_0'' = 100{,}000\underline{/65.00 \text{ deg.}} = 42{,}262 + j\,90{,}631 \text{ volts},$$
$$I = i_0' + ji_0'' = 200.0\underline{/40.00 \text{ deg.}} = 153.208 + j\,128.558 \text{ amp}.$$

Then

$$e_0 = 100{,}000 \text{ volts},$$
$$i_0 = 200.00 \text{ amperes},$$
$$\phi_0 = -25.000 \text{ degrees},$$

and

$$P_0 = e_0'i_0' + e_0''i_0'' = e_0i_0 \cos \phi_0 = 18{,}126{,}300 \text{ watts},$$

$$P_0^j = e_0'i_0'' - e_0''i_0' = e_0i_0 \sin \phi_0 = -8{,}452{,}400 \text{ volt-amps.},$$

$$U = \sqrt{\frac{Z}{Y}} = u_1 + ju_2 = 406.61 - j\,135.63 \text{ ohms},$$

$$\frac{1}{U} = \sqrt{\frac{Y}{Z}} = y_1 + jy_2 = 0.0022131 + j\,0.00073821 \text{ mhos}.$$

From Table XIX,

$$\alpha l = 0.037205,$$

$$\beta l = 5.2264 \text{ degrees}.$$

Substituting these numerical values in equation (12), the four separate terms which go to make up the total power at the generator, P, are, in consecutive order:

First term	+	18,176,700	watts
Second term	+	1,429,810	"
Third term	−	177,520	"
Fourth term	−	84,280	"
$P =$		19,344,710	watts.

Computing the generator e.m.f. and current by equation (19), Chapter VI, we have

$$E = 41{,}892 + j\,98{,}894 \text{ volts},$$

$$I = 132.10 + j\,139.65 \text{ amperes},$$

and the power

$$P = e_1i_1 + e_2i_2 = ei \cos \phi = 19{,}344{,}500 \text{ watts},$$

which verifies completely the value obtained from equation (12)

If the line loss be roughly computed by

$$P' = i_0^2rl + e_0^2gl,$$

we obtain $P' = 1{,}250{,}000$ watts, while the true value of the line loss is

$$P' = P - P_0 = 1{,}218{,}410 \text{ watts},$$

the last three figures of which are unreliable, since the computations were made with five-place logarithm tables.

125. Line Loss. — For very accurate work, it may be desirable to use the formula

$$\begin{aligned} P' = P - P_0 = P_0\, 2 \sinh^2 \alpha l \\ + (e_0^2 y_1 + i_0^2 u_1) \sinh \alpha l \cosh \alpha l \\ - (e_0^2 y_2 + i_0^2 u_2) \sin \beta l \cos \beta l \\ + 2\,(P_0 u_2 y_2 + P_0^j u_1 y_2)\,(\sinh^2 \alpha l + \sin^2 \beta l), \end{aligned} \qquad (13)$$

to calculate the line loss directly, for this obviates the numerical inaccuracy involved in taking the difference between two large and nearly equal quantities, in the case of a line whose losses are small in comparison with the power transmitted. With long telephone circuits, on the other hand, the loss in the line may be many times the power received at the load end, so that values of power loss for a given load may be conveniently obtained either by numerically subtracting, $P - P_0$, from equation (12), or by the use of (13) where the subtraction has been made in the analytical expression.

126. Unloaded Line. — For an unloaded line, $P_0 = 0$, the expression for P is also the expression for line loss, and it is

$$\begin{aligned} P\Big]_{P_0=0} = (e_0^2 y_1 + i_0^2 u_1) \sinh \alpha l \cosh \alpha l \\ - (e_0^2 y_2 + i_0^2 u_2) \sin \beta l \cos \beta l \\ + 2\, P_0^j u_1 y_2\, (\sinh^2 \alpha l + \sin^2 \beta l). \end{aligned} \qquad (14)$$

The equation as given may be applied to a line supplying a load of zero power-factor, so that $P_0 = 0$. P_0^j is then numerically equal to $e_0 i_0$ — positive for a leading load and negative for a lagging load. If both P_0 and P_0^j are zero, then the line must be either short-circuited or open-circuited ($e_0 = 0$ or $i_0 = 0$), and for these conditions, equation (12) becomes

$$\left.\begin{aligned} P\Big]_{e_0=0} = i_0^2\,(u_1 \sinh \alpha l \cosh \alpha l - u_2 \sin \beta l \cos \beta l), \\ P\Big]_{i_0=0} = e_0^2\,(y_1 \sinh \alpha l \cosh \alpha l - y_2 \sin \beta l \cos \beta l). \end{aligned}\right\} \qquad (15)$$

127. Power in Distortionless Lines. — In Chapter VII, the quantity $V = \alpha + j\beta$ was discussed, and by equations (13) and (14) it is seen that for all frequencies, the attenuation constant, α, and the velocity of propagation, S, are constant, and the wave-

length constant, β, is directly proportional to the frequency when the relation

$$gL = rC \text{ (for distortionless line)} \tag{16}$$

is fulfilled. A line whose properties are thus related is said to be a "distortionless line." The reason for this name is quite obvious — the constancy of α and S for all frequencies. If an alternating wave is made up of a number of harmonic frequencies, all of these different frequencies in either an e.m.f. or a current wave will be propagated along such a line with the same attenuation and the same velocity. Hence the wave form of the propagated wave does not change, all of the component harmonics retaining the same relative phase position throughout the line, and all decreasing in the same proportion. This, of course, is true only for the component (main and reflected) waves, and not for the resultant wave, except in the case of an infinitely long line, where the main wave and resultant wave are identical. This matter of wave distortion is of the utmost importance in telephonic work, and will be discussed in connection therewith.

In such a distortionless line, then, by equation (16),

$$\frac{C}{g} = \frac{L}{r} = k,$$

where k is the proportionality constant. Then

$$\left.\begin{aligned} Z &= r + j\omega L = r + j\omega kr, \\ Y &= g + j\omega C = g + j\omega kg, \end{aligned}\right\} \tag{17}$$

so that

$$U = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{r}{g}}\sqrt{\frac{1 + j\omega k}{1 + j\omega k}} = \sqrt{\frac{r}{g}} + j\,0.$$

This equation, interpreted, shows that for a distortionless circuit, the infinite line impedance, U, contains no imaginary component, and has a numerical value equal at all frequencies to the infinite line resistance for direct current, $\sqrt{\frac{r}{g}}$.

$$\left.\begin{aligned} U\Big]_{gL=rC} &= u_1 + ju_2 = \sqrt{\frac{r}{g}}, \\ Y\Big]_{gL=rC} &= y_1 + jy_2 = \sqrt{\frac{g}{r}}, \\ u_2\Big]_{gL=rC} &= 0, \quad y_2\Big]_{gL=rC} = 0. \end{aligned}\right\} \tag{18}$$

Introducing this condition, that both u_2 and y_2 are zero, into the general equation for power, (12), there results, for the distortionless line,

$$P\Big]_{gL=rC} = P_0\,(1 + 2\sinh^2 \alpha l) + \left(e_0^2\sqrt{\frac{g}{r}} + i_0^2\sqrt{\frac{r}{g}}\right)\sinh \alpha l \cosh \alpha l, \tag{19}$$

a very simple and interesting form. Since

$$\cosh 2\,\alpha l = 1 + 2\sinh^2 \alpha l$$

it is seen that equation (19), for a load power-factor of unity, is identical with equation (26) of Chapter II for a direct current line.

In this case, as has been shown to be true for any frequency,

$$\alpha\Big]_{gL=rC} = \sqrt{rg},$$

so that, finally,

$$P\Big]_{gL=rC} = P_0\cosh 2\sqrt{grl} + \frac{1}{2}\left(e_0^2\sqrt{\frac{g}{r}} + i_0^2\sqrt{\frac{r}{g}}\right)\sinh 2\sqrt{rgl}. \tag{20}$$

This resulting value of power at the generator end for a given power, P_0, at the load end, is entirely independent of the frequency, in so far as the fundamental line constants r, L, g, and C are independent of the frequency.

Consider the numerical case used as illustration of equation (12) in the last paragraph. For this line, $gL = 0.0003060 \times 10^{-6}$ and $rC = 0.0040150 \times 10^{-6}$. Suppose g, r, and C to remain constant, but by some means the value of L is increased until the relation, $gL = rC$, is fulfilled. The line then becomes distortionless, and equation (20) applies. The self-inductance, L, must be increased from 0.002040 to 0.0267667 henry per mile to bring about this condition. Then, for the same load for which the preceding numerical calculation was made, we have for the power at the generator, by equation (20),

$$P = 19{,}391{,}400 \text{ watts}$$

and

$$P' = 1{,}265{,}200 \text{ watts} = \text{line loss.}$$

The line loss with the original value of self-inductance per mile was 1,218,400 watts, indicating that, in this case, the creating of

a distortionless circuit by an increase of the self-inductance does not bring about a decrease in the line power loss. From this particular numerical example it is not possible to generalize, however, to the extent of saying that the creation of a distortionless line by an increase in the self-inductance will always cause an increase in the power loss for a given load. It is possible to differentiate equation (13) with respect to either L or C (remembering that the auxiliary constants, u_1, u_2, y_1, y_2, α, and β, are functions of r, g, L, and C), and thus to determine for what values of the independent variable the line loss would be a minimum, but the resulting expression after differentiating would be so complicated as to prevent any practical results from being obtained. For a given load, and either L or C variable, it would be best to plot a curve showing the power loss as a function of the variable, and from the curve to estimate the value for a minimum loss.

128. Load Impedance for a Minimum Line Loss with a Given P_0. — In transmitting power over a leaky line by means of a direct current, a maximum efficiency of transmission occurs when the load-end resistance (equivalent resistance of the load) is equal to the infinite-line resistance $\sqrt{\frac{r}{g}}$, according to equation (34), Chapter II. With this value of load resistance, the efficiency of transmission is constant and independent of the magnitude of voltage and current.

A similar condition obtains in the case of a line carrying alternating currents. As an introductory case, suppose an amount of power, P_0, is to be delivered at the load end of a line, at a power-factor, $\cos \phi_0$. This constant amount of power may be supplied by current at any voltage, the necessary relation being

$$e_0 i_0 \cos \phi_0 = P_0, \quad \text{or} \quad e_0 i_0 = \frac{P_0}{\cos \phi_0}.$$

What must the values of e_0 and i_0 be in order that the line loss be a minimum?

Let $z_0 = \frac{e_0}{i_0}$ = scalar value of load-end impedance.

Then

$$i_0^2 = \frac{P_0}{z_0 \cos \phi_0} \quad \text{and} \quad e_0^2 = \frac{z_0 P_0}{\cos \phi_0}. \tag{21}$$

Since P_0 and ϕ_0 are assumed constant, $P_0{}^j = P_0 \tan \phi_0$ is also a constant.

Thus, if W represent the variable portion of the line loss as e_0 and i_0 are varied, with constant P_0 and ϕ_0, we have, from equation (13),

$$W = \tfrac{1}{2}\,(e_0{}^2 y_1 + i_0{}^2 u_1) \sinh 2\,\alpha l - \tfrac{1}{2}\,(e_0{}^2 y_2 + i_0{}^2 u_2) \sin 2\,\beta l,$$

and by equation (21),

$$W = \frac{P_0}{2}\left(\frac{y_1 z_0}{\cos \phi_0} + \frac{u_1}{z_0 \cos \phi_0}\right) \sinh 2\,\alpha l - \frac{P_0}{2}\left(\frac{y_2 z_0}{\cos \phi_0} + \frac{u_2}{z_0 \cos \phi_0}\right) \sin 2\,\beta l. \tag{22}$$

In order that this variable portion of the line loss, and thus the total line loss, be a minimum, differentiate equation (22) with respect to z_0 and equate the derivative to zero.

$$\frac{dW}{dz_0} = \frac{P_0}{2} \sec \phi_0 \left(y_1 - \frac{u_1}{z_0{}^2}\right) \sinh 2\,\alpha l - \frac{P_0}{2} \sec \phi_0 \left(y_2 - \frac{u_2}{z_0{}^2}\right) \sin 2\,\beta l = 0. \tag{23}$$

Solving (23) for z_0,

$$z_0{}^2 = \frac{u_1 \sinh 2\,\alpha l - u_2 \sin 2\,\beta l}{y_1 \sinh 2\,\alpha l - y_2 \sin 2\,\beta l} = \frac{u_1 \sinh 2\,\alpha l - u_2 \sin 2\,\beta l}{u_1 \sinh 2\,\alpha l + u_2 \sin 2\,\beta l}\, u^2. \tag{24}$$

The value of z_0 given by equation (24) is independent of the amount of power delivered, as well as of the load power-factor. That is, for a given amount of power delivered, to secure a minimum line loss under any condition, the load impedance should have the above numerical value. For varying amounts of delivered power and varying power-factors, such a condition demands, therefore, a variable and varying generator voltage — a condition which is usually not feasible, in power transmission at least. Notice that, in regard to the above statement of minimum line loss under any condition, an absolute minimum of line loss is not implied. For a given amount of power at a given power-factor the line loss is a minimum when z_0 is determined by the above equation, but obviously, by equation (13), for a given power P_0 and load impedance z_0, the line loss may be changed by varying the power-factor angle, ϕ_0, of the load.

129. Numerical Illustration. — As illustration of equation (24) consider the 400 mile, 60 cycle line, whose constants are given in Tables XII and XIV and some operating characteristics of which are given in Tables XV and XVI. A summary of the constants, and the calculations leading to a value of z_0 by equation (24) is given below in Table XXVI.

TABLE XXVI

DETERMINATION OF z_0 FOR MINIMUM LINE LOSS, BY EQUATION (24)

r	0.275 ohm per mile
g	0.15×10^{-6} mho per mile
L	0.00204 henry per mile
C	0.0146 microfarad per mile
l	400 miles
f	60 cycles per second
$2\,\alpha l$	0.312615
$2\,\beta l$	95.5384 degrees
$U = u_1 + ju_2$	$385.143\underline{/-9°\ 03'\ 26''.9} = 380.34 - j\,60.630$
$\frac{1}{U} = y_1 + jy_2$	$0.00259644\underline{/9°\ 03'\ 26''.9} = 0.0025641 + j\,0.00040875$
$\sinh 2\,\alpha l$	0.31773
$\sin 2\,\beta l$	0.99534
$u_1 \sinh 2\,\alpha l$	120.84
$u_2 \sin 2\,\beta l$	−60.35
$y_1 \sinh 2\,\alpha l$	0.00081468
$y_2 \sin 2\,\beta l$	0.00040684
Numerator	181.19
Denominator	0.00040784
z_0^2	444260
z_0	666.53 ohms

It thus appears that, for this line, a minimum line loss will be secured when, for any load, the generator voltage is so adjusted that the ratio, z_0, between the numerical values of load voltage and load current is 666.53. For example, to supply a load of, say, 4000 kilowatts at a power-factor of cos 25 degrees = 0.9063, the load voltage should, by (21), be

$$e_0 = \sqrt{z_0 \frac{P_0}{\cos \phi_0}} = \sqrt{666.53 \frac{4{,}000{,}000}{0.9063}} = 54{,}238 \text{ volts},$$

and the corresponding load current is

$$i_0 = \frac{e_0}{z_0} = 81.373 \text{ amperes.}$$

Assuming this to be a lagging load, we have, with $\phi_0 = -25.000$ degrees and $P_0 = 4{,}000{,}000$ watts,

$$P_0{}^j = P_0 \tan \phi_0 = -1{,}865{,}200 \text{ volt-amperes.}$$

By equation (13),

$$P' = \text{line loss} = 951{,}030 \text{ watts.}$$
$$P = \text{power at generator} = 4{,}951{,}030 \text{ watts.}$$

$$\text{Efficiency of transmission} = \eta = \frac{P_0}{P} = 0.8079, \text{ or } 80.79 \text{ per cent.}$$

Note that this value of efficiency corresponds approximately with that for the 80 ampere entry in the first part of Table XV.

130. Line Efficiency. — As before pointed out, if the load impedance remain fixed, the current and voltage throughout the line change directly as the generator voltage. All powers and volt-amperes then vary as the square of the generator voltage, so that the efficiency remains constant. Since for any value of power, P_0, a minimum line loss, and therefore a maximum efficiency of transmission, is obtained with the above value of z_0, it follows that each particular power-factor of load has a definite and characteristic maximum possible efficiency, viz., that corresponding to the value of z_0 by equation (24). A curve may thus be plotted showing the maximum possible efficiencies as a function of the load power-factor, and such a curve is independent of the line voltage employed.

Let η = efficiency of transmission $= \dfrac{P_0}{P}$ and $Z_0 = z_0\underline{/-\phi_0}$.

The angle of Z_0 is taken negatively because an impedance with a positive angle, $r + jx = z\underline{/\psi}$, corresponds to a negative power-factor angle (lagging) load, and vice versa. In this discussion, ϕ_0 is the power-factor angle — the angle of I_0 with respect to E_0.

Then, by equations (12) and (21),

$$\frac{1}{\eta} = 1 + 2\sinh^2 \alpha l + 2\,(u_2 y_2 + u_1 y_2 \tan \phi_0)\,(\sinh^2 \alpha l + \sin^2 \beta l)$$
$$+ \frac{1}{2\cos\phi_0}\left[\left(y_1 z_0 + \frac{u_1}{z_0}\right)\sinh 2\,\alpha l - \left(y_2 z_0 + \frac{u_2}{z_0}\right)\sin 2\,\beta l\right]. \qquad (25)$$

If ϕ_0 be kept constant in the above equation for efficiency, and the first derivative with respect to z_0 be equated to zero in order to determine the value of z_0 for either a maximum or a minimum, the result given by equation (24) will be obtained. Substituting

(24) for z_0 in equation (25), the efficiency expression for this special load impedance becomes:

$$\left.\frac{1}{\eta}\right]_{z_0 \text{ by } (24)} = 1 + 2\sinh^2 \alpha l + 2\, u_2 y_2 (\sinh^2 \alpha l + \sin^2 \beta l) + 2\, u_1 y_2 (\sinh^2 \alpha l + \sin^2 \beta l) \tan \phi_0 + \frac{1}{\cos \phi_0} \sqrt{u_1 y_1 \sinh^2 2\,\alpha l + u_2 y_2 \sin^2 2\,\beta l}. \tag{26}$$

131. Load Power-Factor for Maximum Efficiency. — Differentiating with respect to ϕ_0,

$$\frac{d\left(\frac{1}{\eta}\right)}{d\phi_0} = 2\, u_1 y_2 (\sinh^2 \alpha l + \sin^2 \beta l) \sec^2 \phi_0 + \sec \phi_0 \tan \phi_0 \sqrt{u_1 y_1 \sinh^2 2\,\alpha l + u_2 y_2 \sin^2 2\,\beta l}.$$

Equating the first derivative to zero, in order to determine ϕ_0 for a maximum or a minimum, and solving,

$$\sin \phi_0 = -\frac{2\, u_1 y_2 (\sinh^2 \alpha l + \sin^2 \beta l)}{\sqrt{u_1 y_1 \sinh^2 2\,\alpha l + u_2 y_2 \sin^2 2\,\beta l}}, \tag{27}$$

for a condition of maximum efficiency, when taken in connection with the value of z_0 by equation (24). To be rigorous, it is necessary to determine that the value of ϕ_0 by (27) corresponds to a minimum of $\frac{1}{\eta}$ and thus to a maximum efficiency, η. For such to be the case, the second derivative of $\frac{1}{\eta}$ with respect to ϕ_0 must be positive for the value of ϕ_0 given by equation (27).

Differentiating again,

$$\begin{aligned}\frac{d^2\left(\frac{1}{\eta}\right)}{d\phi_0^2} &= 2\, u_1 y_2 (\sinh^2 \alpha l + \sin^2 \beta l)\, 2 \sec^2 \phi_0 \tan \phi_0 \\ &\quad + \sqrt{u_1 y_1 \sinh^2 2\,\alpha l + u_2 y_2 \sin^2 2\,\beta l}\, (\sec^3 \phi_0 + \tan^2 \phi_0 \sec \phi_0) \\ &= \sec^3 \phi_0 \left\{4\, u_1 y_2 (\sinh^2 \alpha l + \sin^2 \beta l) \sin \phi_0 \right. \\ &\quad \left. + (1 + \sin^2 \phi_0) \sqrt{u_1 y_1 \sinh^2 2\,\alpha l + u_2 y_2 \sin^2 2\,\beta l}\right\}.\end{aligned}$$

Substituting for $\sin \phi_0$ the value given by (27),

$$\left.\frac{d^2\left(\frac{1}{\eta}\right)}{d\phi_0^2}\right]_{\substack{z_0 \text{ by } (24) \\ \phi_0 \text{ by } (27)}} = \{u_1 y_1 \sinh^2 2\,\alpha l + u_2 y_2 \sin^2 2\,\beta l - 4\, u_1^2 y_2^2 (\sinh^2 \alpha l + \sin^2 \beta l)^2\} \times \frac{\sec^3 \phi_0}{\sqrt{u_1 y_1 \sinh^2 2\,\alpha l + u_2 y_2 \sin^2 2\,\beta l}}. \tag{28}$$

It remains to be demonstrated that the value of the expression in equation (28) is always positive. From purely physical considerations, we know that if there are no line losses at all, the efficiency of transmission will always be unity, and therefore there will be no maximum or minimum and the second derivative as well as the first derivative have zero values. When there are line losses, the efficiency must reach a maximum value somewhere between its two zero values corresponding to leading and lagging loads of zero power-factor, but of the proper impedance z_0 to make equation (28) applicable. Sin ϕ_0 by equation (27) must then be real, from which we know that the quantity under the radical must be positive. The value of the radical itself is essentially positive, since the scalar value of z_0 by (24) must be positive. Sec$^3 \phi_0$ is always positive, so the value of the second factor in (28) is always positive. The quantity in the brackets remains in question.

To show that the quantity in brackets is always positive appears rather complicated, but some special cases covering the widest possible range of conditions possible in transmission lines may be investigated with ease.

For a distortionless line, $gL = rC$, the value obviously is positive, since by (17) u_2 and y_2 both vanish, and u_1 and y_1 are positive.

Again, suppose that $g = 0$ and $L = 0$, a condition representing maximum distortion. Then

$$Z = r + j\,0 \qquad \text{and} \qquad Y = 0 + j\,2\pi fC = 0 + jb.$$

$$U = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{r}{b}}\underline{/-45^\circ} = \sqrt{\frac{r}{2b}}\,(1 - j\,1).$$

$$u_2 = -u_1 = -\sqrt{\frac{r}{2b}}.$$

$$y_2 = y_1 = \sqrt{\frac{b}{2r}}.$$

$$V = \alpha + j\beta = \sqrt{jrb} = \sqrt{rb}\underline{/45^\circ} = \sqrt{\frac{rb}{2}} + j\sqrt{\frac{rb}{2}}.$$

$$\alpha = \beta = \sqrt{\frac{rb}{2}}.$$

Then

$$u_1 y_1 = \tfrac{1}{2}, \qquad u_2 y_2 = -\tfrac{1}{2}, \qquad \text{and} \qquad u_1^2 y_2^2 = \tfrac{1}{4}.$$

Substituting these values in equation (28) and denoting by B the

value of the expression in brackets, we have, when x represents the common value of αl and βl,

$$B = \tfrac{1}{2}(\sinh^2 2x - \sin^2 2x) - (\sinh^2 x + \sin^2 x)^2.$$

Expanding in a series,

$$B = \tfrac{4}{3}x^4 + \tfrac{16}{35}x^8 + \cdots.$$

The coefficients of the series all being positive, the value of B is always positive, for this special case, and therefore the value of the second derivative of $\dfrac{1}{\eta}$ is positive.

If we take the other extreme, that is, $r = 0$ and $C = 0$, the same result will be obtained.

Since for the above three conditions as widely separated as possible the value of ϕ_0 given by equation (27) corresponds to a maximum value of transmission efficiency as desired, it is reasonable to assume that the equation is correct under all conditions. Several numerical examples with intermediate conditions have shown this to be true.

132. Numerical Illustration. Variation of Efficiency with Power-Factor. — Applying equation (27) for ϕ_0 to the numerical case under discussion, see Table XXVI, the computation appears as below, in Table XXVII.

TABLE XXVII

DETERMINATION OF ϕ_0 FOR MAXIMUM EFFICIENCY OF TRANSMISSION

Values from the logarithmic computation for Table XXVI

log $u_1y_1 \sinh^2 2\alpha l$	8.99322–10
* log $u_2y_2 \sin^2 2\beta l$	–8.39008–10
log sum of above	8.86864–10
log denominator	9.43432–10
log numerator	9.25072–10
* log sin ϕ_0	–9.81640–10
ϕ_0	–40.938 degrees
cos ϕ_0	0.75542

* The minus sign before the logarithm does not signify a negative logarithm, but that the number represented thereby is negative.

The angle of Z_0 is of opposite algebraic sign from that of ϕ_0, so that

$$Z_0 = z_0 \cos\phi_0 - jz_0 \sin\phi_0.$$

From Table XXVI, $z_0 = 666.53$ ohms, which gives $Z_0 = 503.51 + j\,436.74$ ohms, for a condition of maximum efficiency of transmission.

It thus appears that, for this line, a maximum efficiency of transmission is obtained when the load power-factor angle is -40.938 degrees; that is, for a lagging load having a power-factor of 75.542 per cent. The numerical value of the load impedance being fixed by equation (24) and Table XXVI, the separate components thereof are determined as indicated in Table XXVII.

$$r_0 = 503.51 \quad \text{and} \quad x_0 = 436.74 \text{ ohms.}$$

With this value of load end impedance, the efficiency always remains constant at its maximum value, regardless of the generator voltage employed, and the power received varies, of course, as the square of the generator e.m.f. The efficiency for the above value of Z_0 is, by equation (25) or (26), 81.550 per cent. Numerically, equation (26) is

$$\frac{1}{\eta} = 1.020869 + 0.17812 \tan \phi_0 + 0.27184 \sec \phi_0,$$

or

$$\eta = \frac{1}{1.020869 + 0.17812 \tan \phi_0 + 0.27184 \sec \phi_0}, \tag{29}$$

from which the following values were obtained:

TABLE XXVIII

TRANSMISSION EFFICIENCY OF 400 MILE, 60 CYCLE TRANSMISSION LINE

$z_0 = 666.53$ ohms. Values from equations (26) and (29)
ϕ_0 = load power-factor angle. η = line efficiency

ϕ_0	η	
deg.		
−90	0.00000	Lagging
−80	0.63444	Lagging
−60	0.79615	Lagging
−40.938	0.81550	Lagging
−40	0.81547	Lagging
−20	0.80301	Lagging
0	0.77357	
20	0.72728	Leading
40	0.65565	Leading
60	0.53388	Leading
80	0.27804	Leading
90	0.00000	Leading

Fig. 67 shows the curves plotted from the above data. It is to be seen that throughout the possible ranges of power-factor, a much better efficiency of transmission is obtained with a lagging load than with a leading load. For the quite common power-factor angle of 40 degrees (P.F. = 0.766), the efficiency of transmission has a value with a lagging load of 81.55 per cent, while at this same power-factor and a leading load, the efficiency is only

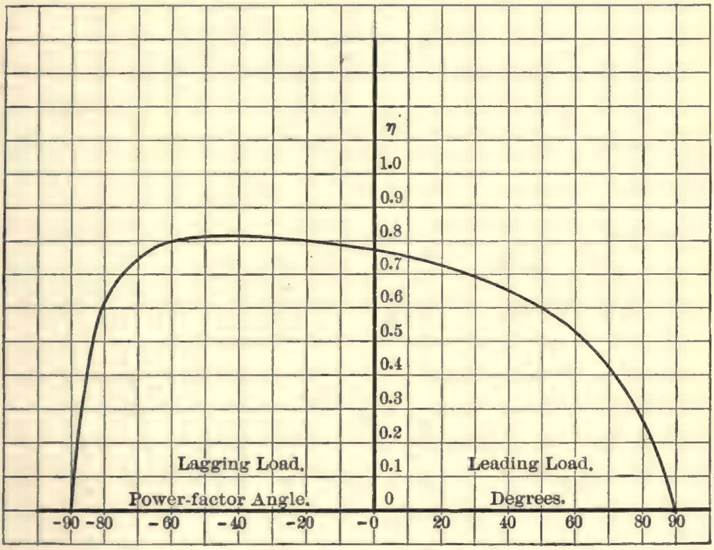

FIG. 67. — Transmission Efficiency with z_0 Constant at its Maximum-efficiency Value. Curve Plotted from Equation (29).

65.56 per cent — a difference of 16 per cent. This being the case, it is obvious that from the standpoint of line losses, or efficiency, a lagging load is much to be preferred to a leading load, although in practice, the latter is viewed with favor on account of the superior voltage regulation obtained thereby. This is a direct result of the use of inherently constant voltage apparatus, and the consequent desirability of maintaining as nearly as may be a condition of constant voltage transmission. Theoretically, the ideal condition of transmission would be that for which the load power-factor would remain constant, while the voltage, as well as the current throughout the line, would vary as the square root of the power transmitted. From such a condition, a maximum economy in the use of power would result.

For this particular line, then, the ideal conditions (as far as efficiency is concerned) may be summarized in a number of equations, as follows:

From equation (24) and Table XXVI,

$$z_0 = 666.53 \text{ ohms.}$$

From equation (27) and Table XXVII,

$$\phi_0 = -40.938 \text{ degrees, and } Z_0 = 503.51 + j\,436.74 \text{ ohms.}$$

From equation (26) and Table XXVIII,

$$\eta = \text{efficiency of transmission} = 0.81550.$$

From equation (21),

$$i_0 = 0.044565\,\sqrt{P_0} \quad \text{and} \quad e_0 = 29.704\,\sqrt{P_0}.$$

At the generator, or sending end,

$$P = \frac{P_0}{\eta} = 1.2263\,P_0.$$

From equation (19), Chapter VI, and Table XIV,

$$i = 0.049348\,\sqrt{P_0} \quad \text{and} \quad e = 32.894\,\sqrt{P_0}.$$

$$i = \sqrt{\frac{1}{\eta}}\,i_0 = 1.1073\,i_0 \quad \text{and} \quad e = \sqrt{\frac{1}{\eta}}\,e_0 = 1.1073\,e_0.$$

From equation (19), Chapter VI,

$$\phi = \text{power-factor angle at the generator} = +40.938 \text{ degrees.}$$

$$z = \frac{e}{i} = \text{apparent generator-end impedance} = 666.53 \text{ ohms.}$$

133. Peculiarities Incident to Maximum-efficiency Transmission. — An inspection of these numerical values reveals the very interesting fact that, for a condition of maximum efficiency, the power-factor angle at the generator end is equal in magnitude, but of opposite algebraic sign from the power-factor angle at the load. The apparent impedance at the generator, z, is numerically equal to the load impedance, z_0. From load to generator, both current and voltage increase by the same percentage, and from the above, the ratio of similar quantities is seen to be

$$\frac{e}{e_0} = \frac{i}{i_0} = \sqrt{\frac{1}{\eta}}.$$

The above statements have been taken from the numerical values for this particular line only, and though the agreement of the numerical values through five significant figures is sufficient to establish their correctness beyond any reasonable doubt, general proof will be given later. (See section 138.)

134. Variation of Maximum-efficiency Conditions with Line Length. — The following table shows the numerical values pertaining to a condition of maximum efficiency of transmission over lines possessing the same fundamental constants per unit length as used in the above numerical illustration and of the various lengths tabulated. The constants are: $f = 60$ cycles, $r = 0.275$, $g = 0.15 \times 10^{-6}$, $L = 0.00204$, and $C = 0.0146 \times 10^{-6}$. For convenience in determining the efficiency for load power-factors other than those corresponding to maximum efficiency, the constants entering into equation (26) are tabulated. They apply to equation (26) when it is written in the form,

$$\frac{1}{\eta} = A + B \tan \phi_0 + C \sec \phi_0.$$

TABLE XXIX

CONDITIONS OBTAINING WHEN LINES OF THE PREVIOUSLY GIVEN CONSTANTS AND OF THE TABULATED LENGTHS ARE OPERATED AT THE MAXIMUM EFFICIENCY POSSIBLE

Miles	z_0, ohms	ϕ_0, degrees	Constants in Eq. 26			η
			A	B	C	
0	1354.0	0	1.0000	0	0	1.00000
100	1243.9	−18.411	1.0008574	0.013788	0.04366	0.95944
200	1023.9	−31.175	1.003812	.052876	.10214	.91641
300	821.3	−38.092	1.009946	.11082	.17964	.86857
400	666.5	−40.938	1.020869	.17812	.27184	.81550
500	552.8	−41.054	1.03845	.24386	.37129	.75848
600	469.7	−39.184	1.06454	.29753	.47091	.69952
800	367.8	−31.066	1.14780	.33942	.65777	.58438
1000	328.4	−19.842	1.27570	.28583	.84210	.48361
1200	334.1	−10.052	1.44355	.18453	1.05731	.40247
1400	365.6	− 4.110	1.64193	.11800	1.30761	.33962
1600	399.9	− 5.431	1.86532	.14971	1.58168	.29071
1800	416.5	− 8.616	2.1188	.28303	1.8892	.25083
2000	409.8	−11.790	2.4183	.45958	2.2493	.21645
Inf.	385.1	− 9.057	Inf.	Inf.	Inf.	0

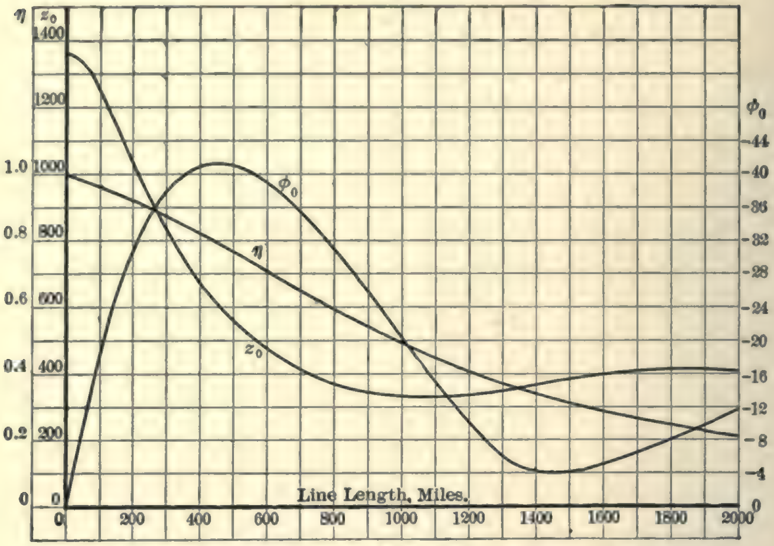

FIG. 68. — Values Obtaining Under Maximum-efficiency Conditions.

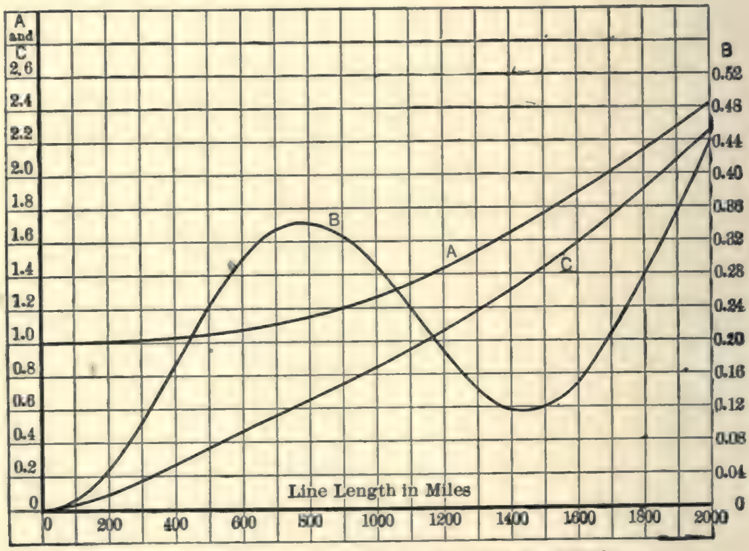

FIG. 69. — The Constants in the Equation for Efficiency,

$$\frac{1}{\eta} = A + B \tan \phi_0 + C \sec \phi_0.$$

Curves plotted from Table XXIX are shown in Figs. 68 and 69. In Fig. 68, for continuously increasing line length, the numerical value of z_0 as shown for a condition of maximum efficiency decreases from a finite maximum at zero length to a finite value at infinite length. On the other hand, ϕ_0, the load power-factor angle for maximum efficiency, increases rapidly in numerical value from zero for a line of zero length to a maximum of about -41 degrees and then decreases, with a two-sided approach to a limiting value for infinite line length. The maximum possible efficiency decreases continuously, approaching zero ultimately.

135. Zero Length Lines. — Special formulæ are required for the determination of the tabulated numerical values for zero line length and infinite line length, for in some cases the general formulæ assume indeterminate forms.

Consider equation (24), for the load-end impedance corresponding to maximum efficiency. For $l = 0$, this equation assumes the form

$$z_0^2 = \frac{0}{0}\, u^2.$$

To evaluate this expression, differentiate both the numerator and the denominator of equation (24) with respect to l for a new numerator and a new denominator, respectively. Thus,

$$z_0^2\Big]_{\substack{P'=\text{min.}\\ l=0}} = u^2\,\frac{\alpha u_1 \cosh 0 - \beta u_2 \cos 0}{\alpha u_1 \cosh 0 + \beta u_2 \cos 0},$$

or

$$z_0\Big]_{\substack{P'=\text{min.}\\ l=0}} = u\sqrt{\frac{\alpha u_1 - \beta u_2}{\alpha u_1 + \beta u_2}}. \tag{30}$$

Further, by substituting

$$\alpha = \sqrt{\tfrac{1}{2}\,(zy - xb + rg)} \quad \text{and} \quad \beta = \sqrt{\tfrac{1}{2}\,(zy + xb - rg)},$$

and also

$$u_1 = \frac{\alpha g + \beta b}{g^2 + b^2}, \quad u_2 = \frac{\beta g - \alpha b}{g^2 + b^2},$$

which follow from the relation

$$U = \sqrt{\frac{Z}{Y}} = \frac{\sqrt{ZY}}{Y} = \frac{V}{Y} = \frac{\alpha + j\beta}{g + jb} = u_1 + ju_2,$$

we obtain as the reduced form of (30),

$$z_0\Big]_{\substack{P'=\text{min.}\\ l=0}} = \sqrt{\frac{r}{g}}. \tag{31}$$

This remarkably simple expression for z_0 applies with considerable approximation to lines of considerable length, as shown by the slow variation of this quantity in the first three entries of Table XXIX. The above value of z_0 is numerically equal to the load-end resistance required for a maximum efficiency of transmission with direct current over a leaky line of any length, for which see equation (34), Chapter II.

136. Infinite Length Lines. — For a line of infinite length, equation (24) may be evaluated by dividing both the numerator and denominator by $\sinh 2\,\alpha l$. For infinite values of l, the fraction $\dfrac{\sin 2\,\beta l}{\sinh 2\,\alpha l}$ vanishes, so that

$$z_0\Big]_{\substack{P'=\text{min.}\\ l=\infty}} = u. \tag{32}$$

Therefore, with a very long line, to secure maximum efficiency, the load-end impedance should have a numerical value equal to u, the impedance of a line of infinite length (see equation (13), Chapter VIII).

For $l = 0$, equation (27) reduces to an indeterminate form. To evaluate, expand both numerator and denominator in series, retaining only the first terms thereof. This yields

$$\sin \phi_0\Big]_{\substack{\eta=\text{max.}\\ l=0}} = -\frac{u_1 y_2\,(\alpha^2+\beta^2)\,l}{\sqrt{u_1 y_1 \alpha^2 + u_2 y_2 \beta^2}}\Bigg]_{l=0} = 0. \tag{33}$$

Thus, for short lines, the maximum efficiency is obtained when the load power-factor angle is small, and for the limiting case of zero length, the angle should be zero.

To evaluate (27) for $l = \infty$, substitute

$$\sinh^2 2\,\alpha l = 4 \sinh^2 \alpha l \cosh^2 \alpha l,$$
$$\sin^2 2\,\beta l = 4 \sin^2 \beta l \cos^2 \beta l,$$

and divide both numerator and denominator by $\sinh^2 \alpha l$.

$$\sin \phi_0\Big]_{\eta=\text{max.}} = -u_1 y_2 \frac{1 + \dfrac{\sin^2 \beta l}{\sinh^2 \alpha l}}{\sqrt{u_1 y_1 \coth^2 \alpha l + u_2 y_2 \dfrac{\sin^2 \beta l \cos^2 \beta l}{\sinh^4 \alpha l}}}. \tag{34}$$

The fractions in both numerator and denominator vanish for infinite values of l, and $\coth^2 \alpha l$ becomes unity. Therefore

$$\sin \phi_0\Big]_{\substack{\eta=\text{max.}\\ l=\infty}} = -\frac{u_1 y_2}{\sqrt{u_1 y_1}} = \frac{u_2}{u} = -\frac{y_2}{y}. \tag{35}$$

In this equation, $\frac{u_2}{u}$ is equal to the sine of the angle, θ_u, of the complex quantity U, which is the infinite-line impedance (see equation (13), Chapter VIII). Combining (32) and (35) we see that for very long (infinite) lines, the load-end impedance for maximum efficiency is

$$Z_0\Big]_{\substack{l=\infty \\ \eta=\max.}} = u\underline{/-\theta_u} = u_1 - ju_2, \tag{36}$$

where $\quad U = \sqrt{\frac{Z}{Y}} = u_1 + ju_2 =$ infinite line impedance.

The imaginary component of Z_0 in equation (36) is of the opposite algebraic sign from the imaginary component of U. This value of Z_0 for maximum efficiency in a very long line is not, as has often been assumed, that for which there is no reflected wave at the load end. From equations (10) and (10b), Chapter VI, the condition necessary in order that the reflected wave be zero is that Z_0 be equal to U, and this is not the same as the condition derived above. Therefore, for maximum efficiency, we can say that, in general, the load impedance must not be such that no reflected wave exists.

137. General. — It is easily seen from equation (27) that, for maximum efficiency, the load power-factor angle, ϕ_0, is always either positive or negative for a line of given fundamental constants, and never varies in algebraic sign as the length of the circuit is changed. The algebraic sign of ϕ_0 is always opposite to that of y_2, and is thus always the same as that of u_2. The angle of the load impedance, Z_0, being equal and of opposite sign to that of ϕ_0, we can say that, in a line of any length, the load impedance for maximum efficiency must always have an angle of opposite sign to the angle of the infinite line impedance, U. In power transmission lines, where the angle of Y is usually greater than the angle of Z, this means that the angle of the load impedance for maximum efficiency is usually positive (the angle of U being negative). Summarized, for maximum efficiency we have:

If $\frac{L}{r} < \frac{C}{g}$, lagging (inductive) load.

If $\frac{L}{r} = \frac{C}{g}$ (distortionless line), non-inductive or unity power factor load.

If $\frac{L}{r} > \frac{C}{g}$, leading (condensive) load.

The above conditions depend only upon the line constants per unit length, and as before mentioned, are independent of the length of line involved.

138. Proof that for Maximum Efficiency the Ratio between e and e_0 Equals the Ratio between i and i_0, and that the Generator Power-factor Angle is Equal in Magnitude but of Opposite Sign to the Load Power-factor Angle. — In section 133, some general conclusions were drawn from a number of numerical relations observed in a particular case. These relations may readily be established as general propositions.

As indicated by equation (16), Chapter II, the solution for a line in which distance is counted positively from the generator towards the load is obtained by substituting $-l$ for l in the equations based on a positive direction counted from load towards generator. Following through the developments leading to the power equation (12) for alternating current circuits, we find the substitution of $-l$ for l to still be permissible. The substitution is also justified in equation (24), where we find the value of z_0 unaffected by the change — both numerator and denominator changing sign. This establishes the fact that, for a minimum line loss with a given power P and power-factor angle ϕ (at the generator), the scalar value of z is equal to the scalar value of z_0. Therefore,

$$\frac{e}{i} = z = z_0 = \frac{e_0}{i_0}, \quad \text{or} \quad \frac{e}{e_0} = \frac{i}{i_0},$$

for maximum efficiency.

Substituting z for z_0, ϕ for ϕ_0, $\frac{1}{\eta}$ for η and $-l$ for l in (25),

$$\eta = 1 + 2\sinh^2 \alpha l + 2\,(u_2y_2 + u_1y_2 \tan\phi)\,(\sinh^2 \alpha l + \sin^2 \beta l)$$
$$- \frac{1}{2\cos\phi}\left\{\left(y_1z + \frac{u_1}{z}\right)\sinh 2\,\alpha l - \left(y_2z + \frac{u_2}{z}\right)\sin 2\,\beta l\right\}.$$

Comparing this equation with (25), we find the coefficient of $\frac{1}{\cos\phi}$ reversed in sign from the coefficient of $\frac{1}{\cos\phi_0}$ in equation (25). The coefficients of $\tan\phi$ and of $\tan\phi_0$ are alike in sign, so that, in equation (27), $\sin\phi$ will be of opposite algebraic sign to $\sin\phi_0$. Such would not appear to be the case from an inspection of the equation for $\sin\phi_0$ alone, for the substitution of $-l$ for l directly in (27) would not reverse the algebraic sign of $\sin\phi_0$. The double

sign should be placed before the radical in equation (27) if + and − values of l are to be used. In the reduction from equation (25) to equation (26), we have for the last term, involving $\cos \phi_0$, by substituting the equation for z_0,

$$+\frac{1}{2\cos\phi_0}\left\{(y_1 \sinh 2\,\alpha l - y_2 \sin 2\,\beta l)\sqrt{\frac{u_1 \sinh 2\,\alpha l - u_2 \sin 2\,\beta l}{u_1 \sinh 2\,\alpha l + u_2 \sin 2\,\beta l}}\,u \right.$$
$$\left. + (u_1 \sinh 2\,\alpha l - u_2 \sin 2\,\beta l)\sqrt{\frac{u_1 \sinh 2\,\alpha l + u_2 \sin 2\,\beta l}{u_1 \sinh 2\,\alpha l - u_2 \sin 2\,\beta l}}\,\frac{1}{u}\right\}.$$

The two radicals, which represent z_0, as well as u are essentially positive, but in combining the expressions in parentheses to obtain the simplified form shown in (26), the algebraic sign of the entire term within the brackets is suppressed, if the positive sign only is retained before the resulting radical. Therefore, for maximum efficiency, $\phi = -\phi_0$, and $\cos\phi = \cos\phi_0$. Then

$$ei \cos\phi = \frac{1}{\eta} e_0 i_0 \cos\phi_0, \quad ei = \frac{1}{\eta} e_0 i_0.$$

Since

$$z = \frac{e}{i} = z_0 = \frac{e_0}{i_0},$$

we have

$$e = \sqrt{\frac{1}{\eta}}\, e_0, \quad i = \sqrt{\frac{1}{\eta}}\, i_0.$$

The deductions drawn from the numerical results in section 132 are thus proved to be true in general.

139. Load Voltage for Maximum All-day Efficiency with Intermittent Load. — If a line is to be operated so as to maintain constant load voltage and carries a given power load for a portion of the time and a negligible load for the remainder of the time (each day), the proper load-end voltage to ensure a minimum total daily loss in the line will be as determined in the following equations.

In a given period of time, T, let t = the time during which full load is carried. Then $T - t$ = time during which there is no load. Throughout the entire time, a constant e.m.f., e_0, is to be maintained at the receiving end. By equation (13), the total energy loss in the line is

$$\begin{aligned} W' = t\,\{&P_0\, 2 \sinh^2 \alpha l + \tfrac{1}{2}(e_0^2 y_1 + i_0^2 u_1) \sinh 2\,\alpha l \\ &- \tfrac{1}{2}(e_0^2 y_2 + i_0^2 u_2) \sin 2\,\beta l \\ &+ 2\,(P_0 u_2 y_2 + P_0' u_1 y_2)(\sinh^2 \alpha l + \sin^2 \beta l)\} \\ &+ (T - t)\,(\tfrac{1}{2} e_0^2 y_1 \sinh 2\,\alpha l - \tfrac{1}{2} e_0^2 y_2 \sin 2\,\beta l). \qquad (37)\end{aligned}$$

The problem is, with a given power-factor angle ϕ_0, and power P_0, what should be the value of e_0 in order that W' be a minimum when taken over a fixed time T?

Placing
$$\frac{t}{T} = k,$$
and substituting,
$$P_0{}^j = P_0 \tan \phi_0, \quad i_0 = \frac{P_0 \sec \phi_0}{e_0}.$$
Then
$$\begin{aligned}\frac{W'}{T} = k\Big\{P_0\, 2 \sinh^2 \alpha l &+ \frac{1}{2}\left(e_0{}^2 y_1 + \frac{1}{e_0{}^2} P_0{}^2 \sec^2 \phi_0 u_1\right) \sinh 2\,\alpha l \\ &- \frac{1}{2}\left(e_0{}^2 y_2 + \frac{1}{e_0{}^2} P_0{}^2 \sec^2 \phi_0 u_2\right) \sin 2\,\beta l \\ &+ 2\,(P_0 u_2 y_2 + P_0 \tan \phi_0 u_1 y_2)\,(\sinh^2 \alpha l + \sin^2 \beta l)\Big\} \\ &+ (1-k)\,\tfrac{1}{2}\,(y_1 \sinh 2\,\alpha l - y_2 \sin 2\,\beta l)\, e_0{}^2. \qquad (38)\end{aligned}$$

Let $\frac{p}{2}$ = variable portion of $\frac{W'}{T}$. The value of e_0 for a minimum of p is to be found.

From equation (38),
$$\begin{aligned}p = e_0{}^2\,&(y_1 \sinh 2\,\alpha l - y_2 \sin 2\,\beta l) \\ &+ \frac{1}{e_0{}^2}\,(u_1 \sinh 2\,\alpha l - u_2 \sin 2\,\beta l)\, k P_0{}^2 \sec^2 \phi_0. \qquad (39)\end{aligned}$$

Differentiating with respect to e_0,
$$\begin{aligned}\frac{dp}{de_0} = 2\,e_0\,&(y_1 \sinh 2\,\alpha l - y_2 \sin 2\,\beta l) \\ &- \frac{2}{e_0{}^3}\,(u_1 \sinh 2\,\alpha l - u_2 \sin 2\,\beta l)\, k P_0{}^2 \sec^2 \phi_0. \qquad (40)\end{aligned}$$

Equating to zero to determine e_0 for a minimum p,
$$e_0 = \left[\frac{(u_1 \sinh 2\,\alpha l - u_2 \sin 2\,\beta l)\, k P_0{}^2 \sec^2 \phi_0}{y_1 \sinh 2\,\alpha l - y_2 \sin 2\,\beta l}\right]^{\frac{1}{4}}. \qquad (41)$$

The above is the desired value of e_0 for which a minimum total energy loss will result. It is seen that the voltage necessary for such a condition varies as the fourth root of the fraction of time, k, during which the power P_0 is to be used. Note that the product $P_0 \sec \phi_0$ is equal to the volt-amperes at the load.

Comparing (41) with (24), we see that if z_0 = load impedance for maximum efficiency as given by (24), we may write

$$e_0 = (z_0^2 k P_0^2 \sec^2 \phi_0)^{\frac{1}{4}}. \tag{42}$$

Taking up the illustrative numerical example in section 129, suppose $k = 0.60$ — that is, the 4000 kilowatts of power is to be used only six-tenths of the time. By the above equation, then, e_0 = 47,736 volts, as compared with 54,238 volts for maximum efficiency when the power is utilized the entire time.

The above development is for a very special type of loading, but in the following a method for determining the proper load voltage for any load distribution will be given.

140. Line Loss with Varying Loads, and the Proper Load Voltage to Secure a Minimum of Such Loss. — Here, we again assume that the receiver, or load, voltage is to be maintained constant throughout all variations of load. The variable load must be given as a function of time, as by a load curve, for instance. We wish to determine what the constant receiver voltage should be in order that the total energy loss in the line, over the period of time in question, will be a minimum.

If W' represents the total energy loss during a given interval of time, by equation (27) and the transformations indicated in (38), during an interval of time, dt,

$$\begin{aligned} dW' = \Big\{ & P_0 \, 2 \sinh^2 \alpha l + \frac{1}{2}\left(e_0^2 y_1 + \frac{P_0^2 \sec^2 \phi_0}{e_0^2} u_1\right) \sinh 2\,\alpha l \\ & - \frac{1}{2}\left(e_0^2 y_2 + \frac{P_0^2 \sec^2 \phi_0}{e_0^2} u_2\right) \sin 2\,\beta l \\ & + 2\,(P_0 u_2 y_2 + P_0 \tan \phi_0 u_1 y_2)\,(\sinh^2 \alpha l + \sin^2 \beta l) \Big\}\, dt. \end{aligned} \tag{43}$$

Rearranging the terms and integrating between the limits 0 and T, where T is the desired interval for which the total energy loss is to be a minimum, under the condition that e_0 remains constant,

$$\begin{aligned} W' = {} & 2\,[(1 + u_2 y_2) \sinh^2 \alpha l + u_2 y_2 \sin^2 \beta l] \int_0^T P_0 \, dt \\ & + 2\, u_1 y_2 (\sinh^2 \alpha l + \sin^2 \beta l) \int_0^T P_0 \tan \phi_0 \, dt \\ & + \tfrac{1}{2} (y_1 \sinh 2\,\alpha l - y_2 \sin 2\,\beta l)\, e_0^2 T \\ & + \tfrac{1}{2} (u_1 \sinh 2\,\alpha l - u_2 \sin 2\,\beta l) \frac{1}{e_0^2} \int_0^T P_0^2 \sec^2 \phi_0 \, dt. \end{aligned} \tag{44}$$

If, at all loads, e_0 is maintained constant, which is the condition for which equation (43) applies, we may differentiate this equation with respect to e_0 in order to determine the proper voltage for a minimum total line loss.

$$\frac{dW'}{de_0} = (y_1 \sinh 2\,\alpha l - y_2 \sin 2\,\beta l)\,Te_0$$
$$- \frac{1}{e_0^3}(u_1 \sinh 2\,\alpha l - u_2 \sin 2\,\beta l)\int_0^T P_0^2 \sec^2 \phi_0\, dt. \quad (45)$$

FIG. 70. — Illustrative Load Curve for 24 Hours.

Equating the above value of the first derivative to zero in order to solve for the value of e_0 which will give a minimum energy loss,

$$e_0 = \left[\frac{(u_1 \sinh 2\,\alpha l - u_2 \sin 2\,\beta l)\int_0^T P_0^2 \sec^2 \phi_0\, dt}{(y_1 \sinh 2\,\alpha l - y_2 \sin 2\,\beta l)\,T}\right]^{\frac{1}{4}}. \quad (46)$$

By equation (24), z_0^2 may be substituted in the above for the fraction made up from the line constants, where z_0 is the load-end impedance corresponding to maximum efficiency of transmission. Thus,

$$e_0\Big]_{W'=\min.} = \sqrt{z_0}\left(\frac{1}{T}\int_0^T P_0^2 \sec^2\phi_0\, dt\right)^{\frac{1}{2}}. \tag{47}$$

$P_0 \sec \phi_0 =$ volt-amperes at the load, and since with a constant load voltage, the current is proportional to the number of volt-amperes, the value of the integral may be taken from charts produced either by a recording ammeter or a recording volt-ampere meter, after new curves, whose ordinates are equal to the squares of the ordinates of the original curves, are plotted. The integral between 0 and T is equal to T times the average value of (volt-amperes)^2.

141. Illustrative Example from Daily Load Curve. — As an illustration of this formula, consider again the 400 mile line which has previously been used. See Table XXVI. Let this be used to supply a load whose value and power-factor for each hour from midnight to midnight are as given in the following table. The quantity, $P_0^2 \sec^2 \phi_0 =$ (volt-amperes)^2, is calculated for each entry and listed in the table. Figure 70 shows the power, power-factor, and (volt-amperes)^2 curves as plotted from Table XXX. The value of the integral may be derived by means of a planimeter, or, since the hourly values are ordinarily connected by straight lines in the load curve, an average of the 24 ordinates will yield the area divided by the base, which is the quantity desired.

From Table XXX,

$$\text{Average value of } (P_0 \sec \phi_0)^2 = \frac{1}{T}\int_0^T P_0^2 \sec^2 \phi_0\, dt$$
$$= 62.8 \times 10^{12} \text{ (volt-amperes)}^2.$$

From Table XXVII, using z_0 for maximum efficiency $= 666.53$ ohms, in equation (47),

$$e_0 = \sqrt{666.53}\,(62.8 \times 10^{12})^{\frac{1}{2}} = 72{,}680 \text{ volts.}$$

This, then, is the proper value of constant load voltage to be used if the total daily energy loss in the line is to be a minimum for the assumed load curve.

TABLE XXX

DATA FOR ILLUSTRATIVE DAILY LOAD, POWER-FACTOR, AND (VOLT-AMPERES)^2 CURVES

Time counted in hours from midnight

t	$P_0 \times 10^{-6}$	$\cos \phi_0$	$P_0^2 \sec^2 \phi_0 \times 10^{-12}$
0	5.0	0.80	39.0
1	4.0	0.75	28.4
2	2.5	0.75	11.1
3	2.2	0.80	7.6
4	2.2	0.80	7.6
5	2.2	0.80	7.6
6	3.1	0.85	13.3
7	6.4	0.85	56.7
8	7.8	0.90	75.2
9	8.2	0.90	83.0
10	7.6	0.92	68.2
11	7.4	0.90	67.6
12	7.4	0.90	67.6
13	7.0	0.92	57.9
14	7.4	0.93	63.2
15	7.6	0.95	63.9
16	8.1	0.96	71.2
17	9.8	0.97	102.0
18	10.0	0.97	106.1
19	9.2	0.95	93.8
20	8.0	0.90	79.0
21	7.3	0.90	65.9
22	6.3	0.87	52.4
23	5.8	0.85	46.5
$T = 24$			1507.0

142. Dependence of Proper Voltage upon the Leakage Coefficient. — The preceding discussions have been illustrated by a line whose dielectric loss coefficient, or conductance, g, is 0.15×10^{-6} mhos per mile. This corresponds to an energy loss of 500 watts per mile at a voltage to neutral of 57,700 volts, or a voltage between wires of a balanced three phase system of 100,000 volts. Since this leakage coefficient is the only line constant of comparatively uncertain value, it may be of interest to ascertain the proper load end impedances for maximum efficiency, as well as the proper load end e.m.f. to give maximum all-day efficiency for the above load curve, when different values of g are assumed.

Table XXXI contains values of the more important quantities entering into the determination of z_0 for maximum efficiency by equation (24) and of the proper voltage e_0 by equation (47) for the load specified in Table XXX, when different values of g are used.

TABLE XXXI

LINE CONSTANTS AND CONDITIONS FOR MAXIMUM EFFICIENCY IN A 400 MILE LINE WITH DIFFERENT VALUES OF g

$g \times 10^6$	0.00	0.05	0.10	0.15	0.20	0.25	0.30
$2\,\alpha l$	0.28981	0.29741	0.30501	0.31262	0.32022	0.32784	0.33545
$2\,\beta l$	95.752	95.682	95.608	95.538	95.468	95.402	95.336
u_1	379.54	379.83	380.10	380.34	380.56	380.75	380.92
u_2	−65.819	−64.093	−62.365	−60.630	−58.897	−57.161	−55.420
z_0	755.25	721.51	692.23	666.53	643.79	623.46	605.29
e_0	77,360	75,610	74,060	72,680	71,430	70,290	69,260

The above values are shown plotted in Fig. 71.

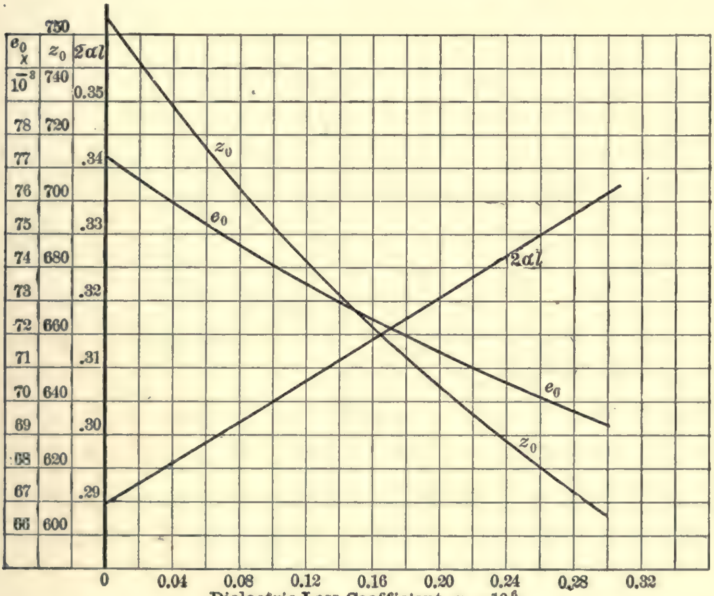

FIG. 71. — Conditions for Maximum All-day Efficiency for the Load Curve of Fig. 70, with Different Values of Leakage g.

143. Determination of the Resistance and Leakage Losses Separately; Graphical Method. — Equation (13) gives the total power lost in the line, which, of course, is made up of the sum of the i^2r and e^2g losses for every element of the line length. As it may sometimes be desirable to compute the losses due to each cause separately, formulæ for this purpose will be given here.

Two methods are available — graphical and analytical — the latter usually being preferable on account of the greater ease in application and the higher accuracy of the results.

To determine the resistance loss for a given load by the graphical method, calculate the line current for a number of equidistant points along the line and plot, in polar coördinates, the curve representing i as a function of line length. The radius, R, in the polar curve is drawn to such a scale as to represent the line current. Suppose a unit angle in our diagram represents h units of line length — angles being counted in radians and distances along the line in whatever units we may select. Let a unit length of radius represent C amperes of current. Then, in the diagram, for any point in the line distant l from the load

$$R = \frac{i}{C}, \quad i = CR \quad \text{and} \quad \theta = \frac{l}{h}, \quad l = h\theta, \tag{48}$$

which are the relations between the coördinates of a point on the polar curve and the quantities in the line. The total resistance loss in the line is

$$P_r = r\int_0^l i^2\,dl, \tag{49}$$

where r is the resistance of the line per unit length (the same unit of length as used in defining h).

The area of the polar curve is

$$A = \int_0 \frac{R^2}{2}\,d\theta. \tag{50}$$

If we substitute (48) in (49), the integration between limits zero and l is replaced by integration between limits 0 and θ, and by this substitution,

$$P_r = rhC^2\int_0^\theta R^2\,d\theta. \tag{51}$$

By equation (50), however, the integral in the above equation represents twice the area of the curve, $2\,A$. Therefore,

$$P_r = \text{resistance loss} = 2\,rhC^2A. \tag{52}$$

In a similar manner, the total leakage loss may be determined by plotting a new polar curve representing the electromotive force at all points along the line. If C_e is the number of volts repre-

sented by a radius of unit length in the diagram, h is the number of units of line length per radian in the diagram, and A_e is the area of the new polar curve,

$$P_g = 2\,ghC_e^2A_e, \tag{53}$$

where g is the conductance of the line per unit length.

144. Illustration of Graphical Method. — As an example, the power losses in the line whose voltage and current distribution are as given in Table XIII may be calculated. In the polar diagrams,

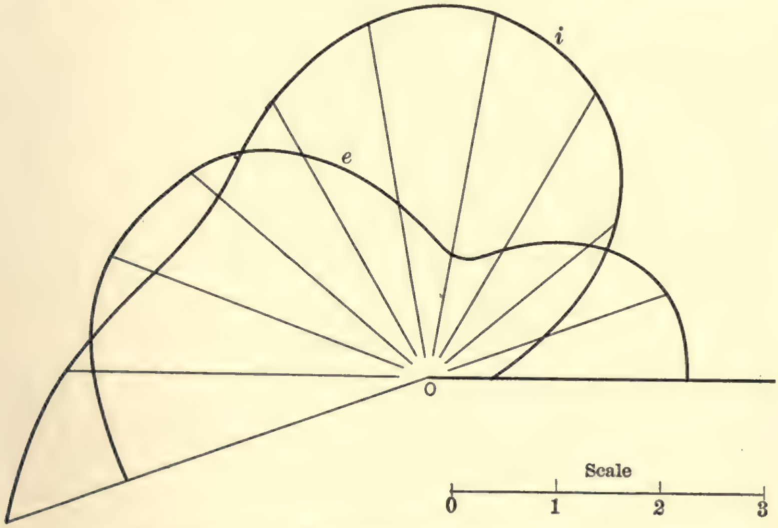

FIG. 72. — Graphical Determination of Separate Power Losses.

or curves, let one degree of angle θ represent ten miles of line. Then

$$h = \frac{360}{2\,\pi}10 = 572.96.$$

Let one inch in the diagram of currents represent 40 amperes, and in the voltage diagram, let one inch represent 20,000 volts. Then $C = 40$ and $C_e = 20{,}000$. For this line, $r = 0.275$ ohm per mile and $g = 0.15 \times 10^{-6}$ mho per mile. Fig. 72 shows the current and voltage diagrams as plotted from the data in Table XIII. From these figures, by the aid of a planimeter,

$A = 15.13$ square inches, and $A_e = 10.52$ square inches.

Therefore,

$P_r = 2 \times 0.275 \times 572.96 \times (40)^2 \times 15.13 = 7{,}629{,}000$ watts.

$P_g = 2 \times 0.15 \times 10^{-6} \times 572.96 \times (20{,}000)^2 \times 10.52 = 723{,}000$ watts.

$P' = P_r + P_g =$ total line loss $= 8{,}352{,}000$ watts.

The above value of P' added to P_0 should equal the total power at the end of the 2000 miles. Forming the sum, we obtain 9,482,000 watts, which differs by 1.4 per cent from the value given in Table XIII. Considering that the quantities are all obtained graphically, the discrepancy is no more than might be expected. Nevertheless, a much closer result is obtained for the mid-point of the line. For this length, the areas of the current and e.m.f. curves are, respectively, 6.27 and 3.12 sq. in. Computing the separate losses and adding to P_0, we obtain 4,510,000 watts for the power at a distance of 1000 miles. The error here is less than one per cent.

At best, the graphical process is difficult to apply on account of the necessity of determining the current and e.m.f. at a sufficient number of points along the line to enable smooth curves to be drawn.

145. Separate Losses Analytically. — To determine the resistance loss analytically, we must form the integral

$$P_r = r \int_0^l i^2\, dl, \tag{54}$$

and this necessitates a knowledge of the square of the scalar value of current at any point in the line. At any point,

$$I = i_1 + ji_2, \tag{55}$$

and in equation (5) we have given the values of i_1 and i_2 in terms of i_0' and i_0'', the two components of the load current. Since the two components of I are at right angles to each other

$$i^2 = i_1^2 + i_2^2. \tag{56}$$

Equation (54) is then

$$P_r = r \int_0 (i_1^2 + i_2^2)\, dl. \tag{57}$$

Substituting (5) in (57), after some transformation,

$$P_r = r\int_0^l \Big\{ i_0^2 \cosh^2 \alpha l \cos^2 \beta l + i_0^2 \sinh^2 \alpha l \sin^2 \beta l$$
$$+ e_0^2 \frac{1}{u^2} \sinh^2 \alpha l \cos^2 \beta l + e_0^2 \frac{1}{u^2} \cosh^2 \alpha l \sin^2 \beta l$$
$$+ 2\, i_0' \,(e_0' y_1 - e_0'' y_2) \cosh \alpha l \sinh \alpha l$$
$$+ 2\, i_0'' \,(e_0' y_2 + e_0'' y_1) \cosh \alpha l \sinh \alpha l$$
$$- 2\, i_0' \,(e_0' y_2 + e_0'' y_1) \cos \beta l \sin \beta l$$
$$+ 2\, i_0'' \,(e_0' y_1 - e_0'' y_2) \cos \beta l \sin \beta l \Big\}\, dl. \tag{58}$$

Introducing the relations,

$$P_0 = e_0' i_0' + e_0'' i_0'' \quad \text{and} \quad P_0^j = e_0' i_0'' - e_0'' i_0',$$

and combining terms,

$$P_r = r\int_0^l \Big\{ \left(i_0^2 + e_0^2 \frac{1}{u^2} \right) \sinh^2 \alpha l$$
$$+ i_0^2 \cos^2 \beta l + e_0^2 \frac{1}{u^2} \sin^2 \beta l$$
$$+ (P_0 y_1 + P_0^j\, y_2) \sinh 2\, \alpha l$$
$$- (P_0 y_2 - P_0^j y_1) \sin 2\, \beta l \Big\}\, dl. \tag{59}$$

Where $\quad \dfrac{1}{u^2} = y_1^2 + y_2^2 = \text{scalar value of } \dfrac{Y}{Z}\cdot$

Integrating,

$$P_r = r \Big[\frac{1}{4\,\alpha} \left(i_0^2 + \frac{e_0^2}{u^2} \right) \sinh 2\, \alpha l$$
$$+ \frac{1}{4\,\beta} \left(i_0^2 - \frac{e_0^2}{u^2} \right) \sin 2\, \beta l$$
$$+ \frac{1}{2\,\alpha} (P_0 y_1 + P_0^j y_2) \cosh 2\, \alpha l$$
$$+ \frac{1}{2\,\beta} (P_0 y_2 - P_0^j y_1) \cos 2\, \beta l \Big]_0^l \cdot \tag{60}$$

Subtracting the value at the lower limit from the value at the upper limit, the final form becomes

$$P_r = r\left\{\frac{1}{4\,\alpha}\left(i_0^2 + \frac{e_0^2}{u^2}\right)\sinh 2\,\alpha l \right.$$
$$+ \frac{1}{4\,\beta}\left(i_0^2 - \frac{e_0^2}{u^2}\right)\sin 2\,\beta l$$
$$+ \frac{1}{\alpha}\left(P_0 y_1 + P_0^j y_2\right)\sinh^2 \alpha l$$
$$\left. - \frac{1}{\beta}\left(P_0 y_2 - P_0^j y_1\right)\sin^2 \beta l\right\} \text{ watts.} \qquad (61)$$

The equation for the total conductance loss may be written by symmetry from equation (61), with the exception that the algebraic signs before the terms containing P_0^j must be reversed. This arises from the fact that, in squaring the expressions for e_1 and e_2 in equation (4), products of the form $i_0'e_0'' - i_0''e_0'$ replace the products of the form $e_0'i_0'' - e_0''i_0'$ as obtained in the above development for P_r.

Thus,

$$P_g = g\int_0^l e^2\,dl = g\left\{\frac{1}{4\,\alpha}\left(e_0^2 + i_0^2u^2\right)\sinh 2\,\alpha l \right.$$
$$+ \frac{1}{4\,\beta}\left(e_0^2 - i_0^2u^2\right)\sin 2\,\beta l$$
$$+ \frac{1}{\alpha}\left(P_0 u_1 - P_0^j u_2\right)\sinh^2 \alpha l$$
$$\left. - \frac{1}{\beta}\left(P_0 u_2 + P_0^j u_1\right)\sin^2 \beta l\right\} \text{ watts.} \qquad (62)$$

Equations (61) and (62) present the desired results. Though they give the losses separately, it must not be assumed that each of these losses is the same as the total line loss which would occur if the other were absent, for in the event of the conductance loss, say, being zero by virtue of a zero value of g, the value of P_r by equation (61) would not be the same as for a condition of g being finite because of the change of α, β, u, y_1, y_2, etc., with the change in g. To determine P_r when $g = 0$, all quantities entering the equation for P_r must be determined for this value of g. This fact seems to have been overlooked by a number of writers who have developed formulæ for power loss in lines.

146. Illustration of Analytical Method. — As an illustration of these equations, let us calculate the separate losses for the line whose constants are given in Table XXVI, when this line supplies a load of 6000 kilowatts under maximum efficiency conditions. By Table XXVII, $z_0 = 666.53$ ohms, and $\phi_0 = -40.938$ degrees, for maximum efficiency. e_0 and i_0 are then computed by equation (21), and $P_0{}^j$ by the equation

$$P_0{}^j = P_0 \tan \phi_0.$$

The main features of the computation appear below in Table XXXII.

TABLE XXXII

CALCULATION OF SEPARATE LINE LOSSES

Equations (61) and (62)

P_0	6,000,000 watts
$P_0{}^j$	−5,204,200 volt-amperes
ϕ_0	−40.938 degrees
e_0	72,760 volts
i_0	109.162 amperes

For the four terms within the brackets

		Resistance loss		Conductance loss	
No. 1		9,677,000		$1{,}435{,}430 \times 10^6$	
2		−2,838,190		$421{,}000 \times 10^6$	
3		835,660		$123{,}960 \times 10^6$	
4		−4,155,100		$616{,}340 \times 10^6$	
	P_r/r	3,519,370	P_g/g	$2{,}596{,}730 \times 10^6$	
	P_r	967,870	P_g	389,510	watts

$$P' = P_r + P_g = 1{,}357{,}380 \text{ watts}$$
$$P = P_0 + P' = 7{,}357{,}380 \text{ watts}$$
$$\eta = \text{Eff.} = \frac{P_0}{P} = 0.81551$$

The above agrees with the value of maximum efficiency given in Table XXVIII as obtained from the efficiency equation (26).

If we compute the resistance loss by the square of the load current times the total line resistance, 1,310,000 watts is the result. The square of the load voltage times the total line conductance gives 317,000 watts. The error in the approximation by this method is considerable.

147. Separate Losses at No Load. — Under no-load conditions, $P_0 = 0$, and $P_0{}^j = 0$. Equations (61) and (62) then become very simple.

SHORT CIRCUIT

$$\left.\begin{aligned} P_r\Big]_{e_0=0} &= \frac{r}{4}\left(\frac{\sinh 2\,\alpha l}{\alpha} + \frac{\sin 2\,\beta l}{\beta}\right) i_0^2, \\ P_g\Big]_{e_0=0} &= \frac{g}{4}\left(\frac{\sinh 2\,\alpha l}{\alpha} - \frac{\sin 2\,\beta l}{\beta}\right) u^2 i_0^2. \end{aligned}\right\} \qquad (63)$$

OPEN CIRCUIT

$$\left.\begin{aligned} P_r\Big]_{i_0=0} &= \frac{r}{4}\left(\frac{\sinh 2\,\alpha l}{\alpha} - \frac{\sin 2\,\beta l}{\beta}\right) \frac{1}{u^2} e_0^2, \\ P_g\Big]_{i_0=0} &= \frac{g}{4}\left(\frac{\sinh 2\,\alpha l}{\alpha} + \frac{\sin 2\,\beta l}{\beta}\right) e_0^2. \end{aligned}\right\} \qquad (64)$$

Series expansions for the above formulæ may be easily obtained, but no advantage would be gained — the above forms lending themselves admirably to the requirements of the computer.

CHAPTER XI

VOLTAGE REGULATION

Case I. Both Components of Load Become Zero Simultaneously

148. Preliminary Developments. — Since most power generation takes place under practically constant voltage conditions, and further, since the major portion of electrical apparatus requires essentially a constant voltage, regard must be taken of the fluctuations of the load-end e.m.f. of a line when the load varies in magnitude. Adjustment as well as regulation of load-end voltage is often accomplished by means of over or under excited synchronous motors, which thereby consume large currents in quadrature with the impressed e.m.f. It becomes necessary, then, to be able to predetermine the magnitudes of such currents required to produce the desired effect.

For any given receiver load, the voltage regulation of a transmission line is defined as,

$$\text{Regulation} = \frac{\text{Rise in voltage from load to no load}}{\text{Load voltage}}, \qquad (1)$$

the generator e.m.f. being maintained constant throughout the change.

At a given load I_0, and load-end voltage E_0, the generator voltage is

$$E = E_0 \cosh Vl + I_0 U \sinh Vl,$$

and if this generator e.m.f. be maintained constant while the load, and therefore I_0, is reduced to zero, the new value of load-end voltage becomes, by equation (8), Chapter IX,

$$E_0' = E \operatorname{sech} Vl = E_0 + I_0 U \tanh Vl. \qquad (2)$$

This equation may be written,

$$E_0 = E_0' - I_0 U \tanh Vl,$$

a form which could have been written immediately from a generalization discussed in a following section.

The difference between the two load-end voltages is

$$E_0' - E_0 = I_0 U \tanh Vl. \tag{3}$$

149. Scalar Values. — This difference divided by E_0 does not give the regulation as defined in equation (1), for the definition is based upon the scalar values of the electromotive forces under the two conditions, while in (3) a vector difference is obtained. Using equation (3) in (1) would yield a complex expression, or value, for the regulation. The numerical value of this complex result would, in general, be greater than the true regulation, since the scalar value of the vector difference between two quantities is greater than the difference between the scalar values of the quantities themselves (unless the two quantities are in phase, when the results coincide). The expressions for line regulation must, then, be based upon scalar values.

150. Introduction of General Form of Equations. — The transmission line equations appear in the form, see equation (31), Chapter VI,

$$\left.\begin{array}{ll} E = AE_0 + BI_0, & E = (a_1 + ja_2)\,E_0 + (b_1 + jb_2)\,I_0, \\ I = AI_0 + CE_0, & I = (a_1 + ja_2)\,I_0 + (c_1 + jc_2)\,E_0, \\ A = \cosh Vl, \quad B = U \sinh Vl, & C = \dfrac{1}{U} \sinh Vl. \end{array}\right\} \tag{4}$$

Under no load,

$$E_0' = \frac{E}{a_1 + ja_2}, \qquad e_0'^2 = \frac{e^2}{a_1^2 + a_2^2} = \frac{e^2}{a^2}. \tag{5}$$

But $E = AE_0 + BI_0$, so that if

$$E_0 = e' + je'', \qquad \text{and} \qquad I_0 = i' + ji'',$$

$$e_0'^2 = \frac{(a_1e' - a_2e'' + b_1i' - b_2i'')^2 + (a_1e'' + a_2e' + b_1i'' + b_2i')^2}{a^2}. \tag{6}$$

By squaring, as indicated, and subtracting $e_0^2 = e'^2 + e''^2$,

$$\begin{aligned} e_0'^2 - e_0^2 = \frac{1}{a^2}\,[&b_1^2i'^2 + b_2^2i''^2 + b_1^2i''^2 + b_2^2i'^2 \\ &+ 2\,(-a_1a_2e'e'' + a_1b_1e'i' - a_1b_2e'i'' - a_2b_1e''i' \\ &+ a_2b_2e''i'' - b_1b_2i'i'' + a_1a_2e'e'' + a_1b_1e''i'' \\ &+ a_1b_2e''i' + a_2b_1e'i'' + a_2b_2e'i' + b_1b_2i'i'')]. \end{aligned} \tag{7}$$

Combining terms, and replacing by equivalents (see sections 122 and 123),

$$e_0'^2 - e_0^2 = \frac{1}{a^2}\{b^2 i_0^2 + 2\,(a_1b_1 + a_2b_2)\,P_0 + 2\,(a_2b_1 - a_1b_2)\,P_0{}^j\}. \quad (8)$$

Referring again to the form of notation described in Chapter X, section 122, where products of the form $u_1v_1 + u_2v_2$ are written $U \cdot V$, and products of the form $u_1v_2 - u_2v_1$ are written $U \times V$, a simplification in the equations may be made.

Thus, equation (8) becomes

$$e_0'^2 - e_0^2 = \frac{1}{a^2}\{b^2 i_0^2 + 2\,A \cdot BP_0 - 2\,A \times BP_0{}^j\}. \quad (8a)$$

151. Reactive Volt-Amperes Required at Load for Zero Regulation. — The above equation gives the difference between the square of the scalar value of the no-load voltage, $e_0'^2$, and the square of the scalar value of the load voltage, e_0^2. If it is desired to so operate the system that the regulation as defined in equation (1) be zero, the amount of reactive power, or volt-amperes, $P_0{}^j$, required to attain such a condition may be determined from equation (8) by placing $e_0'^2 - e_0^2 = 0$. The voltage at no load will then coincide with the load voltage, numerically. i_0^2 may be eliminated from the equation by the relation

$$e_0 i_0 = \sqrt{P_0^2 + P_0^{j2}}, \qquad i_0^2 = \frac{P_0^2 + P_0^{j2}}{e_0^2}. \quad (9)$$

Making this substitution, and solving the resulting quadratic in $P_0{}^j$,

$$P_0{}^j = \frac{e_0^2}{b^2}\left\{a_1b_2 - a_2b_1 \pm \sqrt{(a_1b_2 - a_2b_1)^2 - \frac{b^2}{e_0^2}\left[\frac{b^2}{e_0^2}P_0^2 + 2\,(a_1b_1 + a_2b_2)P_0\right]}\right\} \quad (10)$$

or in simplified form

$$P_0{}^j = \frac{e_0^2}{b^2}\left\{A \times B \pm \sqrt{(A \times B)^2 - \frac{b^2}{e_0^2}\left[\frac{b^2}{e_0^2}\,P_0^2 + 2\,A \cdot BP_0\right]}\right\}. \quad (10a)$$

In the above equation,

$$\left.\begin{aligned} a_1 &= \cosh \alpha l \cos \beta l, \qquad a_2 = \sinh \alpha l \sin \beta l, \\ b_1 &= u_1 \sinh \alpha l \cos \beta l - u_2 \cosh \alpha l \sin \beta l, \\ b_2 &= u_1 \cosh \alpha l \sin \beta l + u_2 \sinh \alpha l \cos \beta l, \\ b^2 &= b_1^2 + b_2^2. \end{aligned}\right\} \tag{11}$$

From these values, the combinations which enter in the equation for P_0^j may be written. Thus,

$$\left.\begin{aligned} A \cdot B = a_1b_1 + a_2b_2 &= u_1 \cosh \alpha l \sinh \alpha l - u_2 \cos \beta l \sin \beta l, \\ A \times B = a_1b_2 - a_2b_1 &= u_2 \cosh \alpha l \sinh \alpha l + u_1 \cos \beta l \sin \beta l, \\ b^2 &= u^2 (\sinh^2 \alpha l + \sin^2 \beta l). \end{aligned}\right\} \tag{12}$$

In computing, values of the combinations only as given by the above formulæ need be determined, and the labor involved is no greater than that required to determine a_1, a_2, b_1, b_2, and b^2 separately.

As may, at times, be found desirable, equation (10) can be expressed in terms of the components of current in phase with and in quadrature with the load voltage e_0. Thus,

$$\left.\begin{aligned} i_0^p &= \text{in phase, or power, component} = \frac{P_0}{e_0}, \\ i_0^j &= \text{quadrature component} = \frac{P_0^j}{e_0}. \end{aligned}\right\} \tag{13}$$

As seen from an inspection of equation (10), two values of P_0^j or of i_0^j are obtained for which zero regulation is secured. For power loads greater than a certain definite amount, depending upon the line properties and the load voltage e_0, the quantity within the radical becomes negative, and therefore P_0^j and i_0^j become imaginary, which means physically that, under such conditions of line and load, it is impossible to secure zero voltage regulation by adding quadrature components of current to the load.

152. Conditions for Minimum Voltage Regulation. — Although under such cases as above it is impossible to secure zero regulation, the quadrature component of the load current may be so adjusted as to secure a minimum voltage regulation. The substitution of (9) in (8) gives

$$e_0'^2 - e_0^2 = \frac{1}{a^2}\left\{ b^2 \frac{P_0^2}{e_0^2} + 2\,A \cdot BP_0 + b^2 \frac{P_0^{j2}}{e_0^2} - 2\,A \times BP_0^j \right\}, \tag{14}$$

from which the change in voltage may readily be computed. To make this change, and therefore the voltage regulation, a minimum, differentiate with respect to $P_0{}^j$ and solve for $P_0{}^j$ after equating the first derivative to zero.

$$\frac{d\,(e_0'^2 - e_0^2)}{dP_0{}^j} = \frac{1}{a^2}\left[2\frac{b^2}{e_0^2}P_0{}^j - 2\,A \times B\right]. \tag{15}$$

Solving the above, after equating to zero,

$$\left.\begin{aligned} P_0{}^j\Big]_{\text{reg.}=\text{min.}} &= \frac{A \times B}{b^2}\,e_0^2 = \frac{u_2 \sinh 2\,\alpha l + u_1 \sin 2\,\beta l}{2\,u^2\,(\sinh^2 \alpha l + \sin^2 \beta l)}\,e_0^2, \\ i_0{}^j\Big]_{\text{reg.}=\text{min.}} &= \frac{A \times B}{b^2}\,e_0 = \frac{u_2 \sinh 2\,\alpha l + u_1 \sin 2\,\beta l}{2\,u^2\,(\sinh^2 \alpha l + \sin^2 \beta l)}\,e_0. \end{aligned}\right\} \tag{16}$$

For power loads below the critical value above which zero regulation cannot be secured, the value of $P_0{}^j$ determined by the above formula will give a negative regulation — that is, the load voltage will be greater than the no-load voltage.

153. Conditions for any Voltage Regulation. — If we so desire, the quadrature component of load current may be so adjusted as to give any arbitrarily selected value to the voltage regulation. Let it be required to determine $P_0{}^j$ to give a specified regulation, R. Then, if e_0 = load voltage and e_0' = no-load voltage,

$$R = \frac{e_0' - e_0}{e_0}, \qquad e_0' = (1 + R)\,e_0. \tag{17}$$

The left hand member of (14) becomes

$$e_0'^2 - e_0^2 = (2\,R + R^2)\,e_0^2. \tag{18}$$

Introducing this value, and solving the quadratic in $P_0{}^j$,

$$P_0{}^j = \frac{e_0^2}{b^2}\left\{A \times B \pm \sqrt{(A \times B)^2 - \frac{b^2}{e_0^2}\left[\frac{b^2}{e_0^2}P_0^2 + 2\,A \cdot BP_0 - a^2(2R + R^2)e_0^2\right]}\right\}. \tag{19}$$

This equation, like (10), may give imaginary values of $P_0{}^j$. With given line properties and load voltage e_0, real values of $P_0{}^j$ will be secured through a greater range in P_0 as independent variable when finite positive values of R are substituted in (19) than could

be obtained from (10). This should be obvious — it really amounts to saying that a larger amount of power can be transmitted with poor voltage regulation than with good. Again, two values of P_0^j, for which the regulation has its specified value, are obtained from the equation.

154. Solutions in Terms of Load Admittance. — If the load admittance Y_0 is given by

$$Y_0 = g_0 + jb_0, \quad \text{then} \quad P_0 = e_0^2 g_0, \quad \text{and} \quad P_0^j = e_0^2 b_0. \quad (20)$$

It is often desirable to set up a solution in terms of b_0 and g_0 rather than in terms of e_0, P_0, and P_0^j for then the solution is independent of the actual voltage employed and of the power transmitted. Substituting (20) in (19),

$$b_0 = \frac{1}{b^2}\left\{A \times B \pm \sqrt{(A \times B)^2 - b^2[b^2 g_0^2 + 2A \cdot B g_0 - a^2(2R + R^2)]}\right\}. \quad (21)$$

Placing $R = 0$, we get the special case given by equation (10),

$$b_0\Big]_{R=0} = \frac{1}{b^2}\left\{A \times B \pm \sqrt{(A \times B)^2 - b^4 g_0^2 - 2b^2 A \cdot B g_0}\right\}. \quad (22)$$

The above values of g_0 and b_0 obtain under load, and must then become zero simultaneously, according to the bases for the development of the above equations. With constant generator voltage and a variable load P_0, a variation of P_0^j according to equation (10) will maintain constant load voltage — P_0^j disappearing simultaneously with P_0, but not being proportional thereto for intermediate loads.

155. Numerical Illustration. — As illustration, consider again the 400 mile, 60 cycle, power line described in Table XXVI. Using the line constants given in the various foregoing tables, equation (21) becomes numerically,

$$10^3 b_0 = 2.1141 \pm \sqrt{4.4693 - 10^6 g_0^2 - 2132.4 g_0 + 5.6057(2R + R^2)}. \quad (23)$$

From this equation the values of b_0 given in the following table were obtained. Both values of b_0 corresponding to particular values of R and g_0 are tabulated. Positive b_0 indicates a condensive (leading) load, and negative b_0 an inductive (lagging) load component.

TABLE XXXIII

VALUES OF b_0 = LOAD SUSCEPTANCE REQUIRED TO GIVE A REGULATION R WITH A LOAD CONDUCTANCE g_0, WHEN BOTH g_0 AND b_0 BECOME ZERO AT NO LOAD. R DEFINED BY EQUATION (1), $b_0 \times 10^3$ TABULATED

R \ g_0	0	0.0002	0.0004	0.0006	0.0008	0.0010	0.0012	0.0014	0.0016	0.0018
0	4.228 0	4.115 0.113	3.973 0.255	3.796 0.432	3.571 0.657	3.270 0.958	2.800 1.428	Imag. Imag.	Imag. Imag.	Imag. Imag.
0.04	4.334 −0.106	4.226 0.002	4.092 0.136	3.927 0.301	3.721 0.508	3.454 0.775	3.077 1.151	Imag. Imag.	Imag. Imag.	Imag. Imag.
0.08	4.438 −0.210	4.336 −0.108	4.209 0.019	4.054 0.174	3.862 0.366	3.621 0.608	3.299 0.930	2.790 1.438	Imag. Imag.	Imag. Imag.
0.12	4.542 −0.314	4.444 −0.216	4.324 −0.096	4.177 0.051	3.998 0.230	3.776 0.452	3.491 0.737	3.089 1.139	Imag. Imag.	Imag. Imag.
0.16	4.645 −0.417	4.551 −0.323	4.436 −0.208	4.298 −0.069	4.129 0.099	3.924 0.305	3.666 0.562	3.323 0.905	2.774 1.455	Imag. Imag.
0.20	4.748 −0.520	4.658 −0.430	4.548 −0.320	4.416 −0.187	4.256 −0.028	4.064 0.164	3.838 0.400	3.525 0.703	3.096 1.132	Imag. Imag.

In Fig. 73 the values of b_0 given in the foregoing table are plotted as functions of g_0. Multiplication of g_0 and b_0 by e_0^2 gives P_0 and P_0^i, respectively. For example, if e_0 = 80,000 volts, and a load of 7680 kilowatts is to be supplied, $g_0 = 0.0012$, and for a regulation of 8 per cent = 0.08, from the tabular data or curves, $b_0 = 0.003299$ or 0.000930. P_0^i then is 21,130 or 5950 kilovolt-amperes. The smaller is the only practical value to use, so the final load power-factor is

$$\cos \phi_0 = \text{P.F.} = \cos \tan^{-1} \frac{5950}{7680} = 0.79, \text{ leading.}$$

156. Nature of Curves Relating g_0 and b_0. — The curves in Fig. 73 are ellipses, with their common center displaced to the left of the Y axis, and above the X axis by an amount $\frac{a_1 b_2 - a_2 b_1}{b^2} = \frac{A \times B}{b^2}$. The standard form for the equation of an ellipse may be derived easily from equation (21) by placing $A \times B$ in the left hand member and then squaring. Proceeding in this way, we obtain as the final form,

$$\left[g_0 + \frac{A \cdot B}{b^2}\right]^2 + \left[b_0 - \frac{A \times B}{b^2}\right]^2 = \frac{a^2}{b^2}(1 + R)^2. \qquad (24)$$

The curves in Fig. 73 plotted from equation (24) are ellipses with unequal axes only because the scales used in plotting g_0 and b_0 are different. For equal scales, the curves become concentric circles, for (24) is the equation of a circle whose radius is

$$\text{Radius} = \frac{a}{b}(1 + R) = \frac{\text{scalar value of } \cosh Vl}{\text{scalar value of } U \sinh Vl}(1 + R). \quad (25)$$

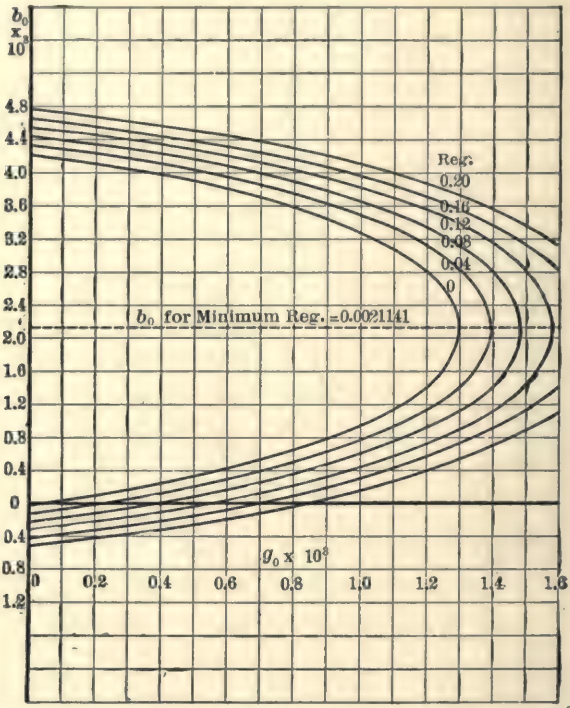

FIG. 73. — Load Susceptance b_0 Required to Give a Regulation R

The coördinates of the center of the system of circles, with reference to the origin for g_0 and b_0, are

$$\left.\begin{aligned} -\frac{A \cdot B}{b^2} &= -\frac{a_1b_1 + a_2b_2}{b^2} = -\frac{1}{2}\,\frac{u_1 \sinh 2\,\alpha l - u_2 \sin 2\,\beta l}{u^2\,(\sinh^2 \alpha l + \sin^2 \beta l)}, \\ \text{and} \qquad & \\ \frac{A \times B}{b^2} &= \frac{a_1b_2 - a_2b_1}{b^2} = \frac{1}{2}\,\frac{u_2 \sinh 2\,\alpha l + u_1 \sin 2\,\beta l}{u^2\,(\sinh^2 \alpha l + \sin^2 \beta l)}. \end{aligned}\right\} \quad (26)$$

157. Geometrical Construction of Curves. — From the information in equations (25) and (26), the entire system of curves may be drawn very quickly with the aid of a compass. First, locate the common center of the circles with respect to the origin of coördinates, and then draw in the circles with radii determined by equation (25), according to the values of regulation, R, desired. For any given load, g_0 and b_0, the regulation may be ascertained from the chart by measuring the distance from the center of the system of circles to the point determined by g_0 and b_0. This length divided by $\frac{a}{b}$ gives $(1 + R)$, and thus R. For the particular problem in hand, the coördinates of the center of the circles are, by (26),

$$-0.0010661 \qquad \text{and} \qquad +0.0021141.$$

By (25), for the radii of the circles we have

$$\begin{aligned}\text{Radius} &= 0.0023677\,(1 + R) \\ &= \text{distance from center of circles to origin} + 0.0023677\,R.\end{aligned}$$

Figure 74 shows the family of circles drawn according to the above data. Reading from the center outwards, the curves are for 0, 0.04, 0.08, 0.12, 0.16, 0.20, and 0.24, values of regulation R. The curves are not numbered, for to do so would crowd the diagram unduly. The data from this set of curves should duplicate that given in Table XXXIII and plotted in Fig. 73.

The power-factor angle of the load may be found graphically from the chart by taking the angle between the g_0 axis and the line joining the point (g_0, b_0) with the origin. Obviously, the scale of the diagram may be so changed as to read in amperes, or in kilowatts and kilovolt-amperes, for any particular value of e_0. Attention may here be called to the fact that this method of curve construction does not apply to transmission line problems only, but to any system for which linear equations of the form

$$E = AE_0 + BI_0$$

apply. Since the general electric circuit is represented by such an equation, the scheme applies thereto. Analytically, then, electric circuits are different only in the manner in which the coefficients A and B are different, in the above type form of equation. As before pointed out, the solution for an electrical

system amounts to the determination of these constants. It is to be noted in equation (25) that the coefficient of $(1 + R)$ is the reciprocal of the scalar value of the short-circuit impedance.

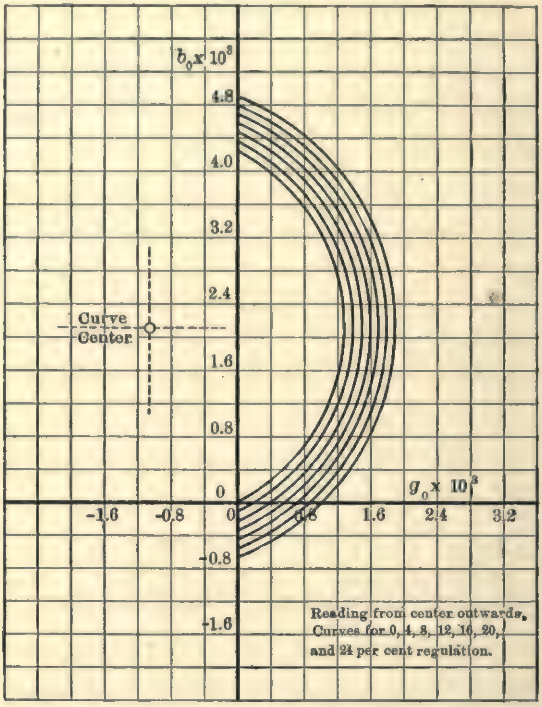

Fig. 74. — Graphical Method for Plotting the Curves Relating g_0 and b_0 for Any Value of R.

158. Relations between Vector Diagrams and Foregoing Curves. — The vector diagram from which data may be taken for the plotting of such curves as Fig. 73 or Fig. 74 is of interest, since it brings out in a clear manner the significance of the graphical construction just described. Fig. 75 is this vector diagram.

If e_0 = load voltage, $e_0' = (1 + R)\, e_0$ = receiver voltage at no load, with regulation R. Since the scalar value of e (at the generator) remains constant, and

$$e = ae_0' = a\,(1 + R)\, e_0,$$

we see that the scalar value of e to produce a regulation R must be given by the above equation. That is, under load, the end of the

vector E must lie on a circle drawn about the origin with a radius $a\,(1 + R)\,e_0$. In Fig. 75, the vector AE_0 is shown, with the family of circles for different values of R.

To the vector AE_0 are added values of Bg_0E_0, for various values of g_0, thus completing the vectorial representation of the equation

$$E = AE_0 + BI_0,$$

for values of $I_0 = (g_0E_0)$ in phase with the load e.m.f. This vector addition gives the points spaced along the line $\overline{AE_0M}$, correspond-

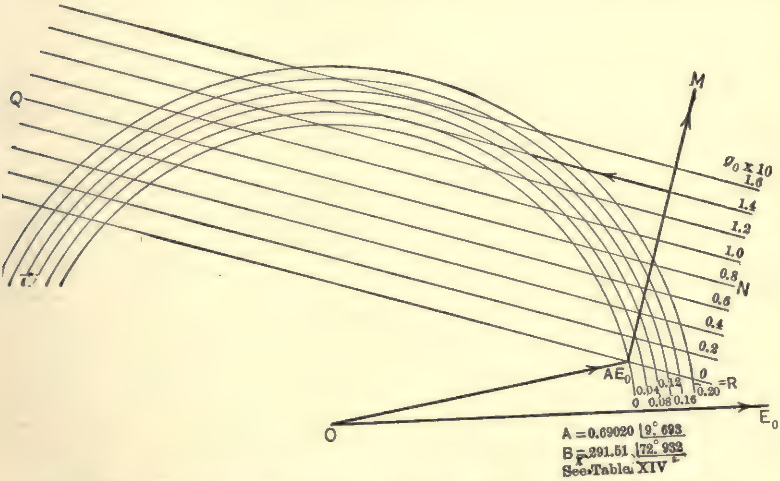

FIG. 75. — Vector Diagram for Regulation Calculations. See Fig. 74.

ing to different values of g_0. A wattless component of load current jb_0E_0 requires the addition of another vector jBb_0E_0 to the vector from the origin to the point on the line $\overline{AE_0M}$, located as described. jBb_0E_0 is at right angles to Bg_0E_0, and therefore to the line $\overline{AE_0M}$. Drawing a line through the point on $\overline{AE_0M}$ which corresponds to some particular value of g_0, we determine the numerical value of the quantity Bb_0E_0 required to give a regulation R by the perpendicular distance from the line $\overline{AE_0M}$ to the ntersections of the new line with the proper circle. These perpendicular distances are the intercepts on the new line, NQ, between the line $\overline{AE_0M}$ and the intersections of NQ with the circles. These intercepts divided by be_0 give the required values of b_0. In the diagram, a number of lines parallel to NQ are drawn, each one corresponding

to a particular value of g_0. By drawing the diagram to be_0 times the voltage scale, distances along the line $\overline{AE_0M}$ represent g_0 directly, and distances along the line NQ represent b_0. The mechanical construction of the curves in Fig. 74 is accomplished by this method. If the above mentioned scale is used, distances along $\overline{AE_0M}$ correspond to distances along the g_0 axis in Fig. 74. The point $\overline{AE_0}$ in Fig. 75 is the origin in Fig. 74, and distances along the lines NQ correspond to distances along the b_0 axis. The b_0 axis itself is represented by the line parallel to NQ drawn through the point $\overline{AE_0}$. The origin in Fig. 75 corresponds to the center of the circles in Fig. 74.

If coördinate paper is used in drawing the curves, as in Fig. 74, the desired data is immediately obtained, without the necessity of actually constructing the vector diagram, Fig. 75.

159. General. — In the above developments it has been assumed that the real power P_0 and the wattless volt-amperes $P_0{}^j$ both become zero at no load, in determining the rise in voltage at the receiving end from load to no-load conditions. When such is the case, $P_0{}^j$ may, as has been shown, be so adjusted as to give zero voltage regulation, according to the definition in (1). If the reactive volt-amperes, $P_0{}^j$, are consumed by a separate machine, or machines, it may happen that P_0 and $P_0{}^j$ would not both become zero, but only the real power. Under such conditions, a different value of $P_0{}^j$ is required to give a specified voltage regulation. If we assume that $P_0{}^j$ remains constant while the real power, P_0, becomes zero, a value of $P_0{}^j$ can be determined for which the regulation will have a certain specified value. Actually, $P_0{}^j$ is made up of two parts, one of which arises from the reactive volt-amperes provided to the load proper, and which, therefore, becomes zero simultaneously with P_0, and the other which consists of the wattless volt-amperes supplied to whatever voltage regulating device (as a synchronous condenser) is permanently connected to the load end of the line, and which, therefore, does not become zero simultaneously with P_0. If we consider the power required to drive the synchronous condenser, the situation becomes still more complicated. When such a condenser is used to consume a wattless component of current, a condition of constant reactive volt-amperes consumed thereby is not secured, for on removing the load, the voltage rises, and on account of this increased voltage, the synchronous condenser will consume a smaller wattless com-

ponent of current, if leading, and a larger wattless component, if lagging. The portion of P_0^j supplied to the synchronous condenser is not constant with varying voltage; neither is the wattless component of current consumed thereby, nor the equivalent susceptance representing the same. To express, analytically, the performance characteristics of such synchronous condensers, and to incorporate these expressions in solutions for line regulation, is scarcely practicable, but in the following, a number of solutions, based upon different assumed load-end conditions, are given.

Case I, for P_0 and P_0^j disappearing simultaneously, has just been given.

Case II. Total Load-end Susceptance Constant

160. Load-end Voltages in Terms of Load-end Admittances. — Here we assume that the total load-end susceptance, b_0, remains constant while the load-end conductance, g_0, becomes zero at no load.

Let

$$\left.\begin{aligned} Y_0 &= g_0 + jb_0 = \text{admittance under load.} \\ Y_0' &= 0 + jb_0 = \text{admittance at no load.} \end{aligned}\right\} \qquad (27)$$

By equation (20), Chapter II, in general, for generator voltage E,

$$E_0 = \frac{E}{\cosh Vl + Y_0 U \sinh Vl} = \text{load voltage,} \qquad (28)$$

and since

$$\cosh Vl = a_1 + ja_2 \quad \text{and} \quad U \sinh Vl = b_1 + jb_2,$$

we have, under load,

$$E_0 = \frac{E}{a_1 + ja_2 + (g_0 + jb_0)(b_1 + jb_2)}$$

and at no load ($g_0 = 0$),

$$E_0' = \frac{E}{a_1 + ja_2 + jb_0(b_1 + jb_2)}. \qquad (29)$$

Let R = voltage regulation, with constant generator voltage, e. Then $e_0' = (1 + R)\,e_0$, and by substituting (29),

$$e_0'^2 = \frac{e^2}{(a_1 - b_0b_2)^2 + (a_2 + b_0b_1)^2} = \frac{(1 + R)^2 e^2}{(a_1 + g_0b_1 - b_0b_2)^2 + (a_2 + g_0b_2 + b_0b_1)^2}$$
$$= (1 + R)^2 e_0^2. \qquad (30)$$

Clearing of fractions, combining terms, and substituting:

$$a_1^2 + a_2^2 = a^2 \quad \text{and} \quad b_1^2 + b_2^2 = b^2,$$

we obtain

$$- b^2 (2R + R^2) b_0^2 + 2(2R + R^2)(a_1 b_2 - a_2 b_1) b_0 \\ + 2(a_1 b_1 + a_2 b_2) g_0 + b^2 g_0^2 - a^2 (2R + R^2) = 0,$$

or, when written in simplified notation,

$$- b^2 (2R + R^2) b_0^2 + 2(2R + R^2) A \times B b_0 \\ + 2 A \cdot B g_0 + b^2 g_0^2 - a^2 (2R + R^2) = 0. \quad (31)$$

161. Constant Susceptance Required for a Given Regulation. — For convenience in writing, let

$$2R + R^2 = m, \qquad m + 1 = (1 + R)^2. \quad (32)$$

Then, by solving (31) for b_0, and combining terms,

$$b_0 = \frac{1}{b^2}\left\{A \times B \pm \sqrt{(A \times B)^2 - a^2 b^2 + 2\frac{A \cdot B}{m} b^2 g_0 + \frac{b^4}{m} g_0^2}\right\}. \quad (33)$$

In equation (33) the constant term within the radical is

$$(A \times B)^2 - a^2 b^2 = -(A \cdot B)^2$$

and therefore, equation (33) reduces to

$$b_0 = \frac{1}{b^2}\left\{A \times B \pm \sqrt{\frac{1}{m}(b^2 g_0 + A \cdot B)^2 - \left(1 + \frac{1}{m}\right)(A \cdot B)^2}\right\}. \quad (34)$$

162. Real and Imaginary Solutions. — In order that b_0 be real, the quantity within the radical must be positive,

$$\frac{1}{m}(b^2 g_0 + A \cdot B)^2 \gtreqless \left(1 + \frac{1}{m}\right)(A \cdot B)^2, \quad (35)$$

$$m + 1 \lesseqgtr \frac{(b^2 g_0 + A \cdot B)^2}{(A \cdot B)^2}, \quad (36)$$

from which, by (32),

$$(1 + R)^2 \lesseqgtr \left(1 + \frac{b^2 g_0}{A \cdot B}\right)^2. \quad (37)$$

Thus, for a real b_0 with a positive R,

$$0 \leqq R \leqq \frac{b^2}{A \cdot B} g_0, \qquad \text{(a)}$$

and for a real b_0 with a negative R, (38)

$$-2 \geqq R \geqq -2 - \frac{b^2}{A \cdot B} g_0. \qquad \text{(b)}$$

No real value of b_0 can be found for a value of R which does not lie within the above limits. In other words, for a given value of g_0, values of R not included within the above limits cannot be produced by shunting g_0 with a constant susceptance, b_0. It is obvious from equation (33) that, for a finite g_0 and $R = 0$, b_0 becomes infinite. Under no condition, then, can zero voltage regulation be secured by such means, though as low a regulation as desired may be secured by the use of a condenser (or inductance, depending on the algebraic sign of b_0 taken from curves similar to Fig. 76) of sufficient size. These facts are emphasized, because in a recent engineering publication a method, based on incorrect ideas, was given for the determination of the constant value of $P_0{}^i$ to give zero regulation. Constant $P_0{}^i$ with zero voltage regulation implies a constant b_0, and according to the above, such a solution is impossible.

163. Numerical Illustration. — Before further development from equation (34), let us see the form of curves resulting therefrom, when applied to the numerical example just considered. Table XXXIV, below, gives values of b_0 computed from (34), which, for the line constants used, becomes

$$10^3 b_0 = 2.1141 \pm \sqrt{\frac{1}{m}(10^3 g_0 + 1.06610)^2 - 1.13657\left(1 + \frac{1}{m}\right)}. \qquad (39)$$

Both values of b resulting therefrom are tabulated.

TABLE XXXIV

VALUES OF b_0 = LOAD SUSCEPTANCE REQUIRED TO GIVE A REGULATION R WITH A LOAD CONDUCTANCE, g_0, WHEN $g_0 = 0$ AT NO LOAD AND b_0 REMAINS CONSTANT. R AS DEFINED BY EQUATION (1). $b_0 \times 10^3$ TABULATED

R \ g_0	0.0002	0.0004	0.0006	0.0008	0.0010	0.0012	0.0014	0.0016	0.0018	0.0020
0	Inf.	Inf.	Inf.	Inf.	Inf.	Inf.	Inf.	Inf.	Inf.	Inf.
0.04	−0.0259	−1.2439	−2.2394	−3.1405	−3.9890	−4.8044	−5.5972	−6.3737	−7.1379	−7.8936
	4.2541	5.4721	6.4676	7.3687	8.2172	9.0326	9.8254	10.6019	11.3661	12.1218
0.08	0.8232	−0.0602	−0.8380	−1.4860	−2.0915	−2.6707	−3.2320	−3.7809	−4.3202	−4.8522
	3.4050	4.2884	5.0662	5.7142	6.3197	6.8989	7.4602	8.0091	8.5484	9.0804
0.12	1.2793	0.4274	−0.1897	−0.7292	−1.2288	−1.7045	−2.1639	−2.6128	−3.0510	−3.4849
	2.9489	3.8008	4.4179	4.9574	5.4570	5.9327	6.3921	6.8410	7.2792	7.7131
0.16	1.6525	0.7746	0.2149	−0.2631	−0.7013	−1.1160	−1.5152	−1.9036	−2.2841	−2.6584
	2.5757	3.4536	4.0133	4.4913	4.9295	5.3442	5.7434	6.1318	6.5123	6.8866
0.20	Imag.	1.0346	0.5050	0.0660	−0.3317	−0.7057	−1.0643	−1.4122	−1.7525	−2.0867
		3.1936	3.7232	4.1622	4.5599	4.9339	5.2925	5.6404	5.9807	6.3149
0.24	Imag.	1.2495	0.7311	0.3178	−0.0514	−0.3962	−0.7252	−1.0436	−1.3541	−1.6590
		2.9787	3.4971	3.9104	4.2796	4.6244	4.9534	5.2718	5.5823	5.8872
0.28	Imag.	1.4433	0.9177	0.5210	0.1725	−0.1502	−0.4567	−0.7524	−1.0403	−1.3224
		2.7849	3.3105	3.7072	4.0557	4.3784	4.6849	4.9806	5.2685	5.5506

164. Nature of Curves. — Plotted, this data gives the family of curves shown in Fig. 76. The curves are hyperbolas which become narrower as R increases. The true shape of the curves is not apparent, since the g_0 and b_0 scales are necessarily taken unequal in order to make the curves readable. From equation (39), it is obvious that the displacement of the axis of the hyperbolas from the g_0 axis is 2.114×10^{-3}. While in the preceding case (g_0 and b_0 disappearing simultaneously) the value $b_0 = 2.1141 \times 10^{-3}$ gives a minimum regulation as pointed out, in this case, where b_0 does not vanish at no load, an inspection of the curves shows that $b_0 = 2.1141 \times 10^{-3}$ is the value which gives a maximum voltage regulation, with a given value of g_0. The left-hand branches of these hyperbolas are not shown, since a negative value of g_0 is meaningless.

Rationalizing equation (34),

$$\left(g_0 + \frac{A \cdot B}{b^2}\right)^2 - (2R + R^2)\left(b_0 - \frac{A \times B}{b^2}\right)^2 = (1 + R)^2\left(\frac{A \cdot B}{b^2}\right)^2. \quad (40)$$

This is the equation of an hyperbola whose center is at the point

$$-\frac{A \cdot B}{b^2}, \qquad +\frac{A \times B}{b^2}. \tag{41}$$

and whose axis is the horizontal line through this point (g_0 being

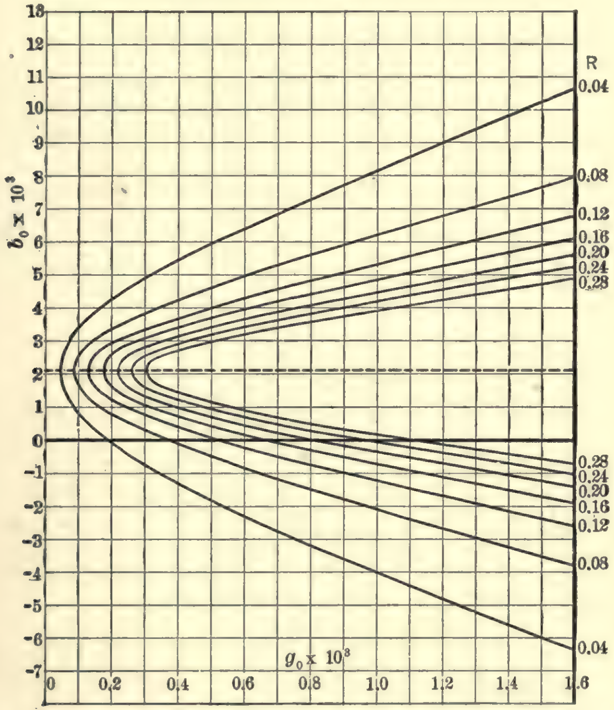

FIG. 76. — Relation between g_0 and b_0 for Any Value of R when b_0 does not Vanish but Remains Constant at No Load. Equation (39).

plotted horizontally). The vertices of the different curves then lie on this line, and their g_0 coördinates are

$$R\,\frac{A \cdot B}{b^2}, \tag{42}$$

for the different values of R. This value may be obtained from equation (38) by using the equality sign, which thus gives the minimum value of g_0 for which a regulation R may be obtained.

The asymptotes to the curves pass through the center (equation 41), with a slope $\frac{1}{\sqrt{2R + R^2}}$, and their equation is, therefore,

$$b_0 - \frac{A \times B}{b^2} = \pm \frac{1}{\sqrt{2R + R^2}} \left(g_0 + \frac{A \cdot B}{b^2} \right). \qquad (43)$$

The substitution of negative values of R according to equation (38) in equation (39) will yield the same family of curves as shown in Fig. 76. The curve for $R = -2.16$, for example, is the same as that for $R = +0.16$, but such solutions are meaningless for a negative regulation numerically greater than 1 has no significance.

165. Significance of the Coördinates of the Central Point. — In this problem, the center for the system of hyperbolas coincides with the center for the system of circles in the foregoing case, and attention is again called to its location in terms of the short-circuit impedance of the line. The general equation on which these discussions are based is

$E = AE_0 + BI_0$, and if we make $E = 0$, we have

$$\frac{E_0}{I_0} = -\frac{B}{A}. \qquad (44)$$

By E_0 is meant the *resulting* terminal e.m.f. at the load end, so that if we consider an e.m.f. *applied* at the load end while the generator end is short circuited ($E = 0$), we have

$$\frac{B}{A} = \frac{\text{applied voltage at load end}}{\text{current produced at load end}}$$
$$= \text{short-circuit impedance measured at load end.} \qquad (45)$$

The reciprocal of this quantity is

$$\frac{A}{B} = \frac{a_1 + ja_2}{b_1 + jb_2} = \frac{a_1b_1 + a_2b_2}{b^2} - j\frac{a_1b_2 - a_2b_1}{b^2} = \frac{A \cdot B}{b^2} - j\frac{A \times B}{b^2}$$
$$= \sqrt{\frac{Y}{Z}} \coth Vl = \text{short-circuit admittance.} \qquad (46)$$

166. Mechanical Construction of Hyperbolas. — The two components of this short-circuit admittance are of opposite algebraic sign to, but the same numerically as the two coördinates of the center of the curves in Figs. 74 and 76. Therefore, if the complex expression for this short-circuit admittance taken nega-

tively, $-\frac{A}{B} = -\sqrt{\frac{Y}{Z}} \coth Vl$, be laid off as a line on the curve sheets on which g_0 and b_0 are to be plotted in the same manner as a vector is drawn in a vector diagram, the center of the curves is at

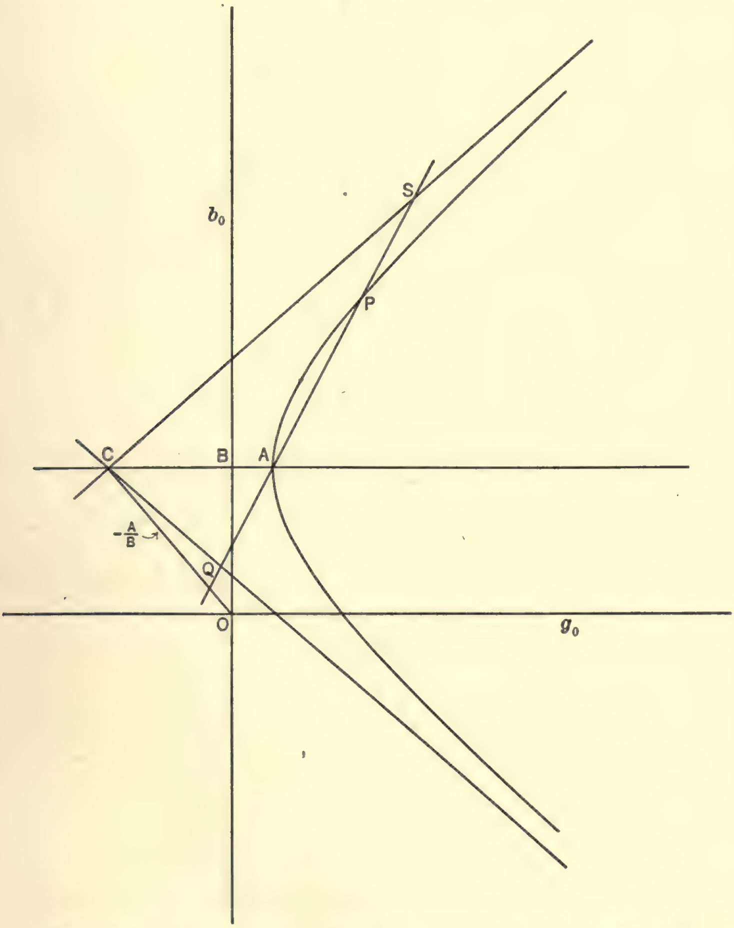

FIG. 77. — Graphical Construction of Curves Shown in Fig. 76.

the end of the line. The circles for the first case may then be drawn according to equation (25), while the hyperbolas which give the solution for this second case may be drawn easily by a graphical process.

Through the curve center, located as above, or by equation (41), draw the straight lines which are the asymptotes of the hyperbola according to equation (43) for any particular value of R. In drawing these asymptotes, it is only necessary to locate one point on each line in addition to the center, C, and this is very easily done by using the known line slope, $\pm \frac{1}{\sqrt{2R + R^2}}$. Account must be taken, of course, of the difference in the g_0 and b_0 scales,

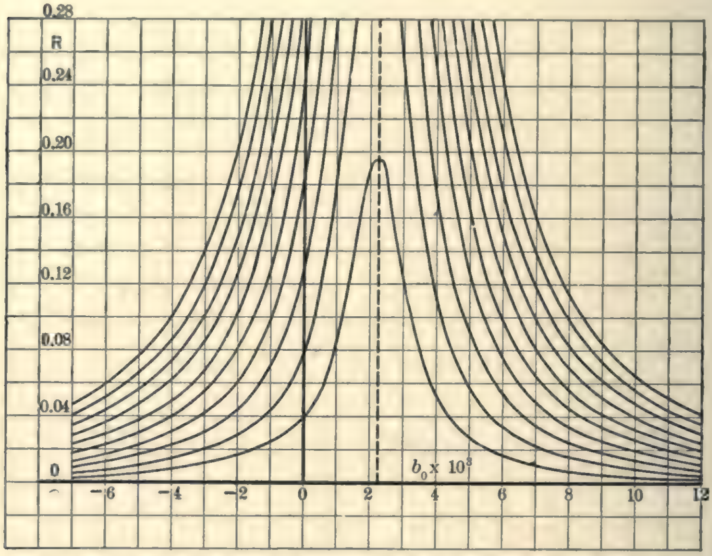

FIG. 78. — Relation between b_0 and R for Different Values of g_0 Cross-plotted from Fig. 76. Reading from Axis of Curves Outward, for g_0 = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, and 2.0 $\times 10^{-3}$ mhos.

if such a difference exists. By equation (42), locate also the vertex of the hyperbola.

Fig. 77 shows the graphical construction of these curves, which, for convenience in reading data, should be drawn immediately upon cross section paper. O is the origin of coördinates, determined by the intersection of the g_0 and b_0 axes. C is the curve center, at the end of the line representing the quantity $-\frac{A}{B}$, the short-circuit admittance taken negatively. The point A being at the vertex of the hyperbola, the distance BA is R times the dis-

tance CB, according to equation (42). After having drawn the asymptotes and located the point A for any value of R in question, draw any line, QS, through the vertex A, and lay off the distance SP equal to AQ. The point P is then a point on the desired hyperbola. Continue in this way until a sufficient number of points are obtained through which to draw the desired curve. This graphical construction remains valid even though the g_0 and b_0 scales are unequal, provided the points C and A are properly located, and the asymptotes drawn according to equation (43).

The family of curves in Fig. 76 could have been drawn by the above method instead of from the numerical data derived by computation from equation (39) and given in Table XXXIV. If the accuracy of plotted curves is sufficient, results may be obtained very rapidly by the use of this graphical process.

By replotting from Fig. 76, the curves shown in Fig. 78 are obtained.

Case III. A Portion Only of the Load-end Susceptance Constant

167. Load-end Voltages. — This case is a continuation in more general form of Case II. Here, the total load end admittance is divided into two parts — the load proper consisting of g_0 and b_0', which make up a load whose power-factor angle is $\phi_0 = \tan^{-1}\dfrac{b_0'}{g_0}$, and a constant load-end susceptance, b_0, which does not disappear simultaneously with the load proper, and which represents approximately a synchronous condenser. The voltage regulation under these conditions is determined by keeping the generator voltage, e, constant in magnitude, as well as b_0, and noting the variation in e_0 when g_0 and b_0' vanish.

Following the same method as in Case II:

Under load, $Y_0 = g_0 + j\,(b_0 + b_0')$,

$$E_0 = \frac{E}{a_1 + ja_2 + (g_0 + jb_0 + jb_0')\,(b_1 + jb_2)}. \tag{47}$$

Under no-load conditions, $Y_0' = 0 + jb_0$, the load-end voltage is

$$E_0' = \frac{E}{a_1 + ja_2 + (0 + jb_0)\,(b_1 + jb_2)}. \tag{48}$$

If R = voltage regulation,

$$e_0'^2 = (1 + R)^2 e_0^2 = \frac{e^2}{(a_1 - b_2 b_0)^2 + (a_2 + b_1 b_0)^2}$$
$$= \frac{(1 + R)^2 e^2}{(a_1 + b_1 g_0 - b_2 b_0' - b_2 b_0)^2 + (a_2 + b_2 g_0 + b_1 b_0' + b_1 b_0)^2}. \quad (49)$$

Simplifying according to the method used with equations (30), (31), and (32), where

$$m = 2R + R^2,$$

$$-mb^2 b_0^2 + 2[b^2 b_0' + m(a_1 b_2 - a_2 b_1)] b_0 + b^2 g_0^2 + 2(a_1 b_1 + a_2 b_2) g_0 + b^2 b_0'^2 - 2(a_1 b_2 - a_2 b_1) b_0' - ma^2 = 0,$$

giving

$$-mb^2 b_0^2 + 2[b^2 b_0' + mA \times B] b_0 + b^2 g_0^2 + 2A \cdot B g_0 + b^2 b_0'^2 - 2A \times B b_0' - ma^2 = 0. \quad (50)$$

168. Constant Portion of Load-end Susceptance for Given Regulation. — Solving the above for b_0, the constant amount of load-end susceptance required to give a regulation R with a load of conductance g_0 and susceptance b_0',

$$b_0 = \frac{b_0'}{m} + \frac{A \times B}{b^2} \pm \frac{1}{m}\sqrt{(1 + m) b_0'^2 + m g_0^2 + 2m \frac{A \cdot B}{b^2} g_0 - m^2 \frac{(A \cdot B)^2}{b^4}}$$
$$= \frac{1}{b^2}\left\{\frac{b^2 b_0'}{m} + A \times B \pm \sqrt{\frac{1 + m}{m^2} b^4 b_0'^2 + \frac{1}{m}(b^2 g_0 + A \cdot B)^2 - \left(1 + \frac{1}{m}\right)(A \cdot B)^2}\right\}. \quad (51)$$

In equation (51), b_0' is the susceptance of the load proper, and when in this expression b_0' is zero, the solution for b_0, the susceptance inserted for purposes of voltage control, corresponds with the solution given in equation (34) for b_0 under such a special condition.

As it stands, equation (51) involves two independent variables, g_0 and b_0', which specify the load conditions, and therefore it is impracticable to plot a sufficient number of curves to cover all of the possibilities therein.

If, in any particular case, the value of b_0' is known for every value of g_0 (as, for example, with a load of constant power-factor, where $b_0' = g_0 \tan \phi_0$), the labor involved in computing a set of data from

which to plot curves similar to those in Fig. 76 is not excessive. The condition of constant power-factor of load proper will be taken up later.

169. Nature and Location of Curves. — Equation (51) assumes the form

$$\left(g_0+\frac{A\cdot B}{b^2}\right)^2-m\left[b_0-\left(\frac{b_0'}{m}+\frac{A\times B}{b^2}\right)\right]^2=(1+m)\left[\left(\frac{A\cdot B}{b^2}\right)^2-\frac{b_0'^2}{m}\right], \quad (52)$$

which is the equation of an hyperbola in the variables g_0 and b_0 when b_0' is constant. The coördinates of the center of the hyperbola are

$$-\frac{A\cdot B}{b^2} \quad \text{and} \quad \frac{b_0'}{2R+R^2}+\frac{A\times B}{b^2}, \quad (53)$$

which are the same as given in equation (41) for a non-inductive load, with the exception of the amount $\frac{b_0'}{2R+R^2}$, which, in this case, is added to the b_0 coördinate of the central point. Changing the value of b_0' thus causes the centers of the curves to take different positions along the perpendicular line passing through the end of the vector $-\frac{A}{B}$. The displacement of the center from this point is

$$\frac{b_0'}{2R+R^2}. \quad (54)$$

The distance from the center to the vertex of the hyperbola is

$$(1+R)\sqrt{\left(\frac{A\cdot B}{b^2}\right)^2-\frac{b_0'^2}{2R+R^2}}, \quad (55)$$

and, as in the preceding case, the asymptotes to the curve have slopes

$$\pm\frac{1}{\sqrt{2R+R^2}}. \quad (56)$$

170. Mechanical Construction of the Curves. — Equations (54), (55), and (56) enable the hyperbolas to be plotted by the convenient graphical construction for any fixed values of R and b_0', though in this case the center of the curve changes its position with each change in the above quantities, according to equations (53) and (54).

The value of the radical in equation (55) may conveniently be found from the simple right-angled triangle relation. Draw two lines intersecting at right angles, and from their intersection lay off $\frac{b_0'}{\sqrt{2R + R^2}}$ along one of them, and with a radius $\frac{A \cdot B}{b^2}$ describe an arc intersecting the other. The intercept is then the desired value of the radical.

Fig. 79 shows the relations existing in the diagram giving b_0 as a function of g_0, for fixed values of R and b_0'. Having the asymptotes

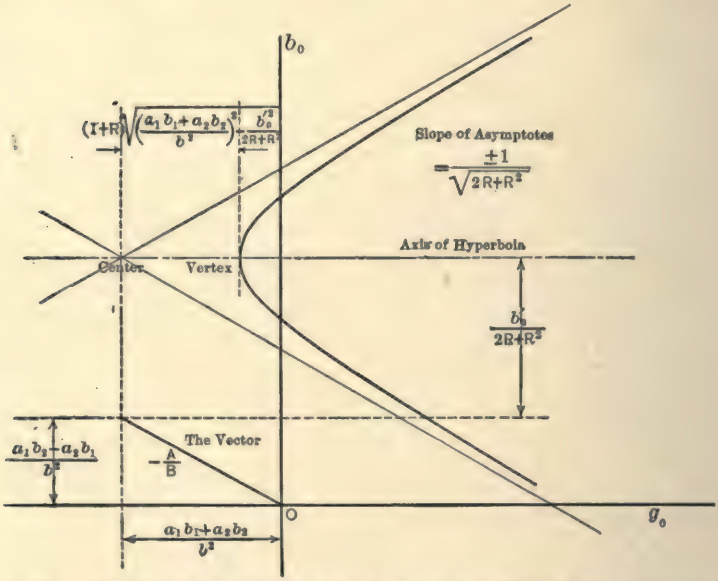

FIG. 79. — Graphical Construction for Plotting the Hyperbolas Represented by Equations (51) and (52).

and the vertex of the hyperbola as obtained by the construction indicated in this figure, the curve may be completed by the mechanical process previously described, Fig. 77.

In any given case, the most convenient procedure probably would be to plot on one curve sheet a number of such curves for different values of b_0' = susceptance of load proper, but all for a fixed value of regulation R. For all of the curves, the asymptotes would then have equal slopes — the center and vertex only changing with changes in b_0'. By cross-plotting from a number of such

curve sheets, each of which is constructed for a different value of R, almost any desired information may be obtained. Fig. 80 shows such a set of curves applying to the same transmission line as treated by Fig. 76, for the particular case, $R = 0.16$. The hyperbolas for $b_0' = 0.0006$, 0.0004, 0.000, -0.0004 and -0.0006 are drawn. The curve for $b_0' = 0.000$ is the same as that shown in Fig. 76 for $R = 0.16$, the difference in their appearance arising from a difference in the scales used in the two curves.

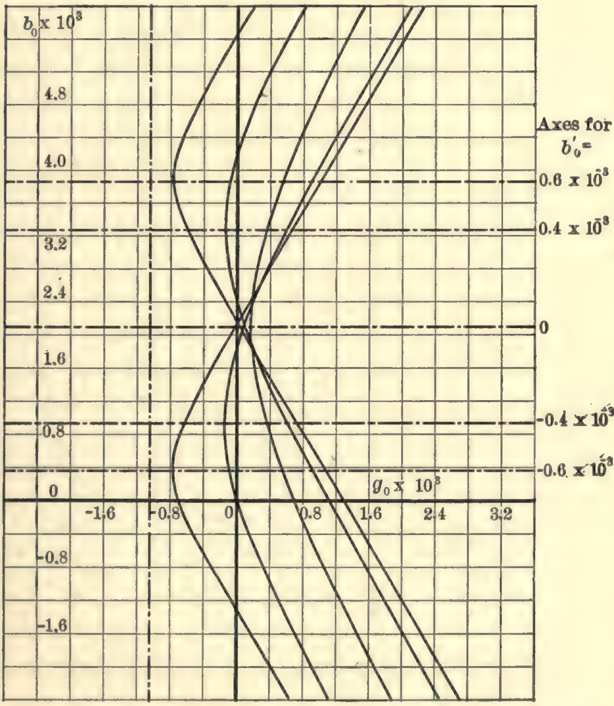

FIG. 80. — Relation between g_0 and b_0 for $R = 0.16$ and for Different Values of b_0'. Locate the Different Curves by Their Axes, which are Numbered with the Corresponding Values of b_0'.

171. Forms of Curves in Special Cases. — As is to be noted from Fig. 80, the hyperbolas have greater curvatures at their vertices for the larger numerical values of b_0'. By equation (55), when

$$b_0'^2 = (2R + R^2)\left(\frac{A \cdot B}{b^2}\right)^2, \tag{57}$$

the distance from the center to the vertex of the hyperbola is zero, and therefore the curve reduces to two intersecting straight lines — the asymptotes. For this particular case, the b_0 coördinate of the center of the curve is

$$\frac{A \times B}{b^2} \pm \frac{1}{\sqrt{2R + R^2}} \frac{A \cdot B}{b^2}, \tag{58}$$

from equation (53), and according to whether b_0' is condensive or inductive the algebraic sign in (58) is positive or negative.

For values of b_0' greater than that indicated by equation (57) the right-hand member of equation (52) becomes negative, and the equation thereby represents hyperbolas whose major axes coincide with the line passing perpendicularly through the end of the vector $-\frac{A}{B}$. The centers of the hyperbolas are located as before, according to equation (53), but the vertices, now falling upon the perpendicular line through the center, are at distances

$$\pm \frac{1 + R}{2R + R^2} \sqrt{b_0'^2 - (2R + R^2)\left(\frac{A \cdot B}{b^2}\right)^2} \tag{59}$$

from the central points. The right-hand sides of both branches of these hyperbolas must be drawn in order to furnish complete information by means of the double value of b_0 secured for each value of g_0, while in the case of the hyperbolas whose major axes are horizontal, the double value of b_0 is secured from both sides of the right-hand branches.

In the particular numerical problem previously used for illustration, where $-\frac{A}{B} = -0.0010661 + j\,0.0021141$, by equation (57) the hyperbolas reduce to straight lines for

$$b_0' = \pm 0.0010661 \sqrt{2R + R^2}.$$

Taking $R = 0.16$ as in Fig. 80, $b_0' = \pm 0.000627$ mho for such a condition. In Fig. 81 the curves for $b_0' = 0.000$, 0.0004, 0.000627, and 0.0010 are shown.

172. Zero Voltage Regulation. — In this case — constant susceptance of load proper — a condition of zero voltage regulation can be secured as long as b_0' has a finite value. The graphical process of plotting the curves cannot be applied for $R = 0$ because the centers of the curves fall at an infinite distance above the g_0

axis and the asymptotes have slopes of $\pm$ infinity. Introducing into equation (50) the condition that $m = 2R + R^2 = 0$, and solving for b_0',

$$b_0\Big]_{R=0} = \frac{A \times B}{b^2} - \frac{b_0'}{2} - \frac{A \cdot B}{b^2}\frac{g_0}{b_0'} - \frac{g_0^2}{2\,b_0'}, \tag{60}$$

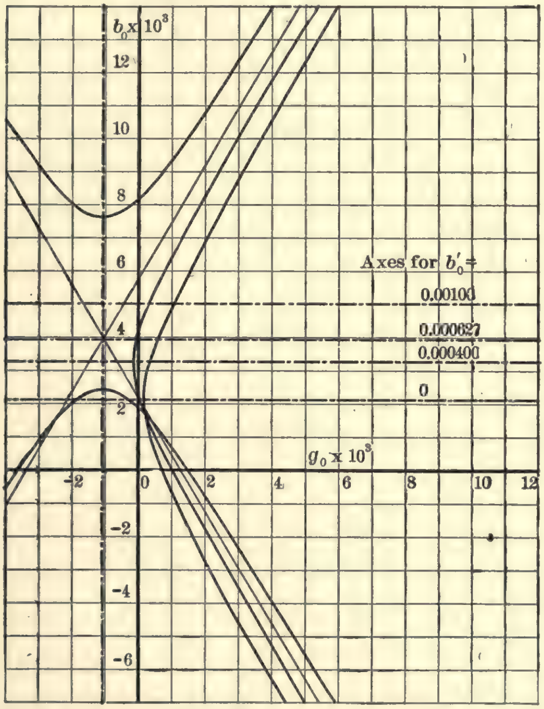

FIG. 81. — Relation between g_0 and b_0 for $R = 0.16$ and Different Values of b_0'. The Curves Become Straight Lines for $b_0' = 0.000627$.

the equation of a parabola in g_0 and b_0. Numerical data for curve plotting is so easily obtained from the above equation that graphical methods of construction need not be developed. For every real value of g_0 as well as for every real value of b_0' a real solution for b_0 is obtained. No imaginary solutions, indicating impossible physical conditions, are to be obtained.

In general, when negative values of R are considered, corresponding to a decrease in load-end voltage when the load proper

is removed, it is easily seen by equation (52) that the curve relating g_0 and b_0 is an ellipse — throughout certain ranges in the value of R. Being of lesser importance, detailed development of this matter is not given here.

Case IV. The Load Proper of Constant Power-Factor

173. Equation Relating Constant Susceptance and Regulation. — Under this head is considered a load proper the two components of whose admittance are always in a constant ratio to each other. The power-factor angle of the load is then constant, and the problem is to determine the proper constant amount of additional susceptance at the load end to produce certain values of voltage regulation.

Let

$$\left.\begin{array}{l} \phi_0 = \text{power-factor angle of load proper, counted positive} \\ \qquad \text{for a leading load.} \\ k = \tan\phi_0 = \dfrac{b_0'}{g_0}, \qquad b_0' = kg_0. \end{array}\right\} \quad (61)$$

g_0 and b_0' are the two components of the load admittance. The above value of b_0' may be substituted directly in equation (50), giving

$$(1 + k^2)\, g_0^2 + 2\, kg_0 b_0 - mb_0^2 + 2\left(\frac{A \cdot B}{b^2} - k\frac{A \times B}{b^2}\right) g_0$$
$$+ 2\, m\frac{A \times B}{b^2}\, b_0 - m\frac{a^2}{b^2} = 0. \qquad (62)$$

174. Location of Curves. — (62) is a general equation of the second degree in g_0 and b_0 as variables, and it therefore represents as a curve some conic section whose center is displaced from the origin of coördinates and whose axis is angularly displaced from the coördinate axes.

From the coefficients of this equation and by the methods of analytical geometry we find that the major axis of the conic section or curve makes an angle, α, with the g_0 axis (counted counter clockwise) determined by

$$\tan 2\,\alpha = \frac{2\,k}{1 + m + k^2}, \quad \alpha = \frac{1}{2}\tan^{-1}\frac{2\,k}{(1 + R)^2 + k^2}. \qquad (63)$$

The center of the curve is displaced from the origin of coördinates by the amounts

$$\left.\begin{aligned} &-\frac{2R+R^2}{(1+R)^2k^2+2R+R^2}\,\frac{A\cdot B}{b^2} \\ \text{and}\quad & \\ &\frac{A\times B}{b^2}-\frac{k}{(1+R)^2k^2+2R+R^2}\,\frac{A\cdot B}{b^2} \end{aligned}\right\} \qquad (64)$$

along the g_0 and b_0 axes, respectively.

175. Nature of Curves. — The nature of the curve representing equation (62) is determined as follows:

$$(1+R)^2k^2+2R+R^2 \begin{cases} <0, \text{ an ellipse,} \\ =0, \text{ a parabola,} \\ >0, \text{ an hyperbola.} \end{cases} \qquad (65)$$

Since negative values of R are seldom considered and since k enters into these determining conditions only as k^2, the left-hand member in the expression (65) is usually positive and greater than zero. The curve is then usually an hyperbola, and graphical methods of construction of the curve for this case only will be considered. To determine the slopes of the asymptotes to the hyperbola it is only necessary to determine the slopes of the curve represented by equation (62) at infinity. Since g_0 and b_0 have infinite values at an infinite distance from the origin, the first powers of these variables as well as the constant term in the equation (62) may be neglected in comparison with their squares and product.

Thus, as approximation at infinity,

$$(1+k^2)\,g_0^2+2\,kg_0b_0-mb_0^2=0,\quad g_0^2+\frac{2\,k}{1+k^2}g_0b_0-\frac{m}{1+k^2}\,b_0^2=0. \qquad (66)$$

Assuming this equation to consist of the product of two linear factors, as

$$\left.\begin{aligned} (g_0+\lambda b_0)\,(g_0+\gamma b_0)&=0, \\ g_0^2+(\lambda+\gamma)\,g_0b_0+\lambda\gamma b_0^2&=0, \end{aligned}\right\} \qquad (67)$$

and equating coefficients of like terms in equations (66) and (67),

$$\left.\begin{aligned} \lambda&=\frac{k+\sqrt{k^2+m\,(1+k^2)}}{1+k^2}, \\ \gamma&=\frac{k-\sqrt{k^2+m\,(1+k^2)}}{1+k^2}. \end{aligned}\right\} \qquad (68)$$

From the two equations,

$$g_0 + \lambda b_0 = 0 \qquad \text{and} \qquad g_0 + \gamma b_0 = 0,$$

and the relation, $m = 2R + R^2$, the slopes of the asymptotes are determined as

$$\begin{aligned} \text{Slopes of asymptotes} &= -\frac{1 + k^2}{k \pm \sqrt{k^2 + m(1 + k^2)}} \\ &= -\frac{1 + k^2}{k \pm \sqrt{(1 + R)^2 k^2 + 2R + R^2}}. \qquad (69) \end{aligned}$$

176. Mechanical Construction. — Knowing, from equation (64), the position of the center of the curve and from equation (69) the slopes of the asymptotes, the hyperbola is completely determined if the position of one point on the curve is known. Two such points may be determined by placing $b_0 = 0$ in equation (62) and solving for g_0.

Thus, for $b_0 = 0$,

$$\begin{aligned} g_0\Big]_{b_0=0} = \frac{1}{(1 + k^2)\, b^2}\Big\{ &-A \cdot B + kA \times B \\ &\pm \sqrt{(A \cdot B - kA \times B)^2 + m(1 + k^2)\, a^2 b^2}\Big\}. \qquad (70) \end{aligned}$$

From the data in equations (63), (64), (69), and (70), the hyperbola may be constructed by the graphical or mechanical process described in Fig. 77, and the labor involved is not excessive.

In this connection it is well to note that the axis and asymptotes should be drawn first, for then it may be possible to determine by inspection whether the two points on the curve, as located by equation (70), will be suitably situated for an accurate continuation of the mechanical process of curve construction. If the two points so located should happen to fall very near to the asymptotes (as compared with the distances from the points to the center) accurate construction by the previously described method is not readily obtained. In such cases, a value of b_0 equal approximately to the ordinate of the center of the curves may be substituted in equation (62) and the corresponding values of g_0 determined by solving the resulting quadratic equation. These points will generally fall near the vertices of the curves — the ideal condition as regards the accuracy of the graphical process of curve drawing.

177. Numerical Illustration. — Figure 82 shows the hyperbola giving the relation between g_0 and b_0 for the numerical case under discussion, for the arbitrarily selected conditions: $R = 0.16$ and $k = \tan\phi_0 = -0.75$, which corresponds to a lagging load of 0.80

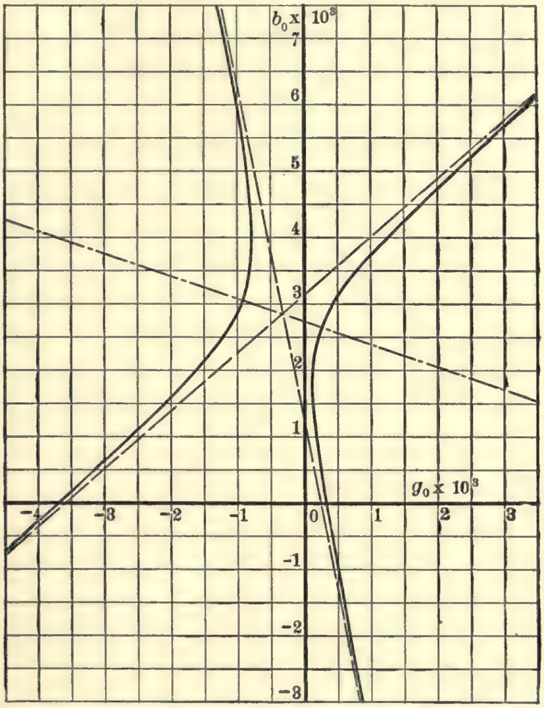

FIG. 82. — Load-end Susceptance b_0 Required for a Regulation of 0.16 at a Constant Power-Factor of Load of 0.80, Lagging.

power-factor. The data from which the curves may be constructed are:

Angle, α, between axis of curves and g_0 axis $= -19.086$ deg.
Angles between asymptotes of curves and g_0 axis, $+40.959$ deg. and -79.131 deg.
Coördinates of center, -0.0003342 and $+0.002839$.
g_0 for $b_0 = 0$, -0.0037268 and $+0.0003327$.
g_0 for $b_0 = 0.00300$, -0.0009578 and $+0.0004437$.

In this particular case the positions of the points as determined for $b_0 = 0$ were not suitable for the process of graphical con-

struction. The positions of the two points for $b_0 = 0.00300$ were therefore computed, and the points so located used as a basis for further construction.

The foregoing methods for the determination and construction of voltage regulation curves for transmission systems are merely illustrative of general methods which may be employed. In regard to the determination of the proper load-end susceptance to give specified values of regulation, a number of special conditions have been considered, but it is obviously impossible to take up individually the great number of such conditions which can arise. In a subsequent volume, dealing primarily with power transmission, a more complete discussion is contemplated. It is obvious that similar developments covering the transmission of power by means of constant, or nearly constant, current are possible.

APPENDIX

HYPERBOLIC FUNCTIONS

A description of the hyperbolic functions with their geometrical significance is not required in order to lead to a clear understanding of their use in the analytical expressions of the solutions of physical problems. For such purposes, the definitions

$$\left.\begin{aligned} \cosh x &= \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \cdots, \\ \sinh x &= \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{6} + \frac{x^5}{120} + \cdots, \\ \tanh x &= \frac{\sinh x}{\cosh x} = x - \frac{x^3}{3} + \frac{2x^5}{15} - + \cdots, \end{aligned}\right\} \qquad (1)$$

usually suffice. Familiarity with the types of differential equations whose solutions are expressible in terms of these functions is desirable. Fig. 83 shows the curves representing the hyperbolic functions of a real variable, and as is apparent from an inspection of the series for these functions, their nature is essentially more simple than that of the trigonometric functions, since they are not periodic.

An excellent working table o the natural values and logarithms of these functions is given by Becker and Van Orstrand in the "Smithsonian Mathematical Tables." The values are given to five decimal places. A collection of formulæ relating to the hyperbolic functions is also given by them, and on account of its usefulness it is, by the permission of the publishers, reproduced here in part. A very useful table of the hyperbolic functions of complex variables has recently been published by Professor A. E. Kennelly. It is particularly useful in the rapid determination of approximate solutions, but for very precise work, the double interpolations involved render its application cumbersome.

As before stated, a five-place accuracy in the tabulated values of functions is sufficient for the majority of physical and engineering calculations, but since a table of hyperbolic functions is not

used as frequently as tables of trigonometric functions or common logarithms, it appeared desirable to the writer to prepare and publish herewith a six-place table of the logarithms of hyperbolic functions. They provide for the exceptional case which requires the degree of precision afforded by logarithmic computations to

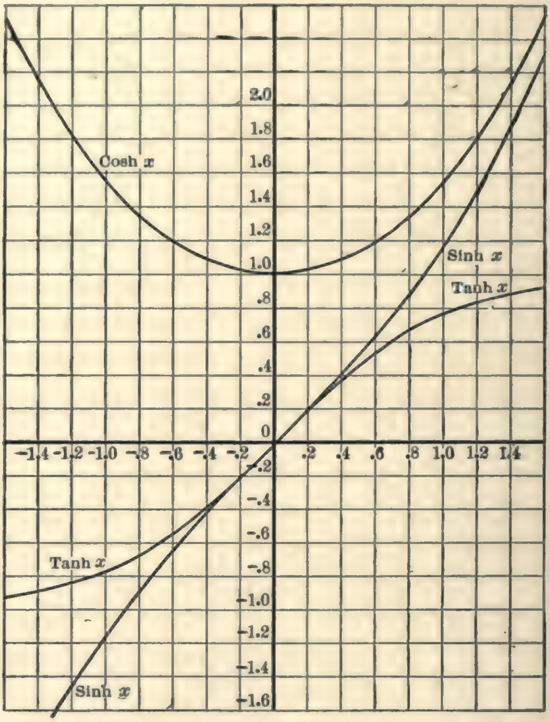

FIG. 83.—The Hyperbolic Functions of a Real Variable.

six significant figures, and at the same time suffice, without an undue increase in the labor of interpolation, for five-place computations.

The six-place tables which follow were taken from twelve-place tables which were computed by B. M. Woods and the writer, and which appear in the Engineering Series of Publications of the University of California. Tabulations of $\log_{10}\cosh x$, $\log_{10}\sinh x$, $\log_{10}\tanh x$, $\log_{10}\dfrac{\sinh x}{x}$, and $\log_{10}\dfrac{x}{\tanh x}$ are given for

values of x from 0.000 to 2.000, which range is ample for the greater portion of engineering problems of the nature treated in this work. Tabular intervals of 0.001 in the argument and the use of the quantities $\log \frac{\sinh x}{x}$ and $\log \frac{x}{\tanh x}$ render interpolation easy in all portions of the table.

The quantities $\log \frac{\sinh x}{x}$ and $\log \frac{x}{\tanh x}$ are given for values of x from 0.000 to 0.500, and serve for the determination of $\log \sinh x$ and $\log \tanh x$ throughout this range where otherwise the interpolations would be troublesome. Thus, given $\log x$ and x:

$$\left.\begin{aligned} &\log \sinh x = \log x + \log \frac{\sinh x}{x}, \\ \text{and} \qquad &\\ &\log \tanh x = \log x - \log \frac{x}{\tanh x}. \end{aligned}\right\} \tag{2}$$

Given $\log \sinh x$ or $\log \tanh x$, interpolations by proportional parts may be made for $\log \frac{\sinh x}{x}$ and $\log \frac{x}{\tanh x}$, since the tabular differences for the latter quantities are small. Thus,

$$\left.\begin{aligned} &\log x = \log \sinh x - \log \frac{\sinh x}{x}, \\ \text{and} \qquad &\\ &\log x = \log \tanh x + \log \frac{x}{\tanh x}. \end{aligned}\right\} \tag{3}$$

The method of use of the auxiliary tables is identical with that of the "S and T" tables used for the determination of the logarithms of the trigonometric sines and tangents of small angles.

In the portion of the table from $x = 0.500$ to $x = 2.000$, the differences between successive values are tabulated, and it is to be noted that these differences are such that linear interpolations only are required.

Relationships Involving Hyperbolic and Allied Functions

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A. — Relations between Hyperbolic and Circular Functions

1. $\sinh u = -i \sin iu = \tan gd\, u.$
2. $\cosh u = \cos iu = \sec gd\, u.$
3. $\tanh u = -i \tan iu = \sin gd\, u.$
4. $\tanh \frac{1}{2} u = \tan \frac{1}{2} gd\, u.$
5. $e^u = (1 + \sin gd\, u) \div \cos gd\, u$
   $= [1 - \cos (\frac{1}{2}\pi + gd\, u)] \div \sin (\frac{1}{2}\pi + gd\, u)$
   $= \tan (\frac{1}{4}\pi + \frac{1}{2} gd\, u).$
6. $\sinh iu = i \sin u.$
7. $\cosh iu = \cos u.$
8. $\tanh iu = i \tan u.$
9. $\sinh (u \pm iv) = \pm i \sin (v \mp iu)$
   $= \sinh u \cos v \pm i \cosh u \sin v.$
10. $\cosh (u \pm iv) = \cos (v \mp iu)$
    $= \cosh u \cos v \pm i \sinh u \sin v.$
11. $\cosh (mi\pi) = \cos m\pi.$ (m is an integer.)
12. $\sinh (2m + 1) \frac{1}{2} i\pi = i \sin (2m + 1) \frac{1}{2}\pi.$ (m is an integer.)

B. — Relations among the Hyperbolic Functions

13. $\sinh u = \frac{1}{2}(e^u - e^{-u}) = -\sinh(-u) = (\operatorname{csch} u)^{-1}$
    $= 2 \tanh \frac{1}{2} u \div (1 - \tanh^2 \frac{1}{2} u) = \tanh u \div (1 - \tanh^2 u)^{\frac{1}{2}}.$
14. $\cosh u = \frac{1}{2}(e^u + e^{-u}) = \cosh(-u) = (\operatorname{sech} u)^{-1}$
    $= (1 + \tanh^2 \frac{1}{2} u) \div (1 - \tanh^2 \frac{1}{2} u) = 1 \div (1 - \tanh^2 u)^{\frac{1}{2}}.$
15. $\tanh u = (e^u - e^{-u}) \div (e^u + e^{-u}) = -\tanh(-u)$
    $= (\coth u)^{-1} = \sinh u \div \cosh u = (1 - \operatorname{sech}^2 u)^{\frac{1}{2}}.$
16. $\operatorname{sech} u = \operatorname{sech}(-u) = (1 - \tanh^2 u)^{\frac{1}{2}}.$
17. $\operatorname{csch} u = -\operatorname{csch}(-u) = (\coth^2 u - 1)^{\frac{1}{2}}.$
18. $\coth u = -\coth(-u) = (\operatorname{csch}^2 u + 1)^{\frac{1}{2}}.$
19. $\cosh^2 u - \sinh^2 u = 1.$
20. $\sinh \frac{1}{2} u = \sqrt{\frac{1}{2}(\cosh u - 1)}.$
21. $\cosh \frac{1}{2} u = \sqrt{\frac{1}{2}(\cosh u + 1)}.$
22. $\tanh \frac{1}{2} u = (\cosh u - 1) \div \sinh u$
    $= \sinh u \div (1 + \cosh u) = \sqrt{(\cosh u - 1) \div (\cosh u + 1)}.$

23. $\sinh 2u = 2 \sinh u \cosh u = 2 \tanh u \div (1 - \tanh^2 u).$

24. $\cosh 2u = \cosh^2 u + \sinh^2 u = 2\cosh^2 u - 1$
$= 1 + 2\sinh^2 u = (1 + \tanh^2 u) \div (1 - \tanh^2 u).$

25. $\tanh 2u = 2 \tanh u \div (1 + \tanh^2 u).$

26. $\sinh 3u = 3 \sinh u + 4 \sinh^3 u.$

27. $\cosh 3u = 4 \cosh^3 u - 3 \cosh u.$

28. $\tanh 3u = (3 \tanh u + \tanh^3 u) \div (1 + 3 \tanh^2 u).$

29. $\sinh nu =$

$$n \cosh^{n-1} u \sinh u + \frac{(n)(n-1)(n-2)}{6} \cosh^{n-3} u \sinh^3 u + \cdots.$$

30. $\cosh nu = \cosh^n u + \dfrac{n(n-1)}{2} \cosh^{n-2} u \sinh^2 u + \cdots.$

31. $\sinh u + \sinh v = 2 \sinh \frac{1}{2}(u + v) \cosh \frac{1}{2}(u - v).$

32. $\sinh u - \sinh v = 2 \cosh \frac{1}{2}(u + v) \sinh \frac{1}{2}(u - v).$

33. $\cosh u + \cosh v = 2 \cosh \frac{1}{2}(u + v) \cosh \frac{1}{2}(u - v).$

34. $\cosh u - \cosh v = 2 \sinh \frac{1}{2}(u + v) \sinh \frac{1}{2}(u - v).$

35. $\sinh u + \cosh u = (1 + \tanh \frac{1}{2} u) \div (1 - \tanh \frac{1}{2} u).$

36. $(\sinh u + \cosh u)^n = \cosh nu + \sinh nu.$

37. $\tanh u + \tanh v = \sinh (u + v) \div \cosh u \cosh v.$

38. $\tanh u - \tanh v = \sinh (u - v) \div \cosh u \cosh v.$

39. $\coth u + \coth v = \sinh (u + v) \div \sinh u \sinh v.$

40. $\coth u - \coth v = - \sinh (u - v) \div \sinh u \sinh v.$

41. $\sinh (u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v.$

42. $\cosh (u \pm v) = \cosh u \cosh v \pm \sinh u \sinh v.$

43. $\tanh (u \pm v) = (\tanh u \pm \tanh v) \div (1 \pm \tanh u \tanh v).$

44. $\coth (u \pm v) = (\coth u \coth v \pm 1) \div (\coth v \pm \coth u).$

45. $\sinh (u + v) + \sinh (u - v) = 2 \sinh u \cosh v.$

46. $\sinh (u + v) - \sinh (u - v) = 2 \cosh u \sinh v.$

47. $\cosh (u + v) + \cosh (u - v) = 2 \cosh u \cosh v.$

48. $\cosh (u + v) - \cosh (u - v) = 2 \sinh u \sinh v.$

49. $\tanh \frac{1}{2}(u + v) = (\sinh u + \sinh v) \div (\cosh u + \cosh v).$

50. $\tanh \frac{1}{2}(u - v) = (\sinh u - \sinh v) \div (\cosh u + \cosh v).$

51. $\coth \frac{1}{2}(u + v) = (\sinh u - \sinh v) \div (\cosh u - \cosh v).$

52. $\coth \frac{1}{2}(u - v) = (\sinh u + \sinh v) \div (\cosh u - \cosh v).$

53. $\dfrac{\tanh u + \tanh v}{\tanh u - \tanh v} = \dfrac{\sinh (u + v)}{\sinh (u - v)}.$

54. $\dfrac{\coth u + \coth v}{\coth u - \coth v} = - \dfrac{\sinh (u + v)}{\sinh (u - v)}.$

55. $\sinh (u + v) + \cosh (u + v) = (\cosh u + \sinh u)(\cosh v + \sinh v).$

56. $\sinh(u+v)\sinh(u-v) = \sinh^2 u - \sinh^2 v$
$= \cosh^2 u - \cosh^2 v.$

57. $\cosh(u+v)\cosh(u-v) = \cosh^2 u + \sinh^2 v$
$= \sinh^2 u + \cosh^2 v.$

58. $\sinh(mi\pi) = 0.$ (m is an integer.)

59. $\cosh(mi\pi) = (-1)^m.$

60. $\tanh(mi\pi) = 0.$

61. $\sinh(u + mi\pi) = (-1)^m \sinh u.$

62. $\cosh(u + mi\pi) = (-1)^m \cosh u.$

63. $\sinh(2m+1)\frac{1}{2}i\pi = \pm i.$

64. $\cosh(2m+1)\frac{1}{2}i\pi = 0.$

65. $\sinh\left(\frac{i\pi}{2} \pm u\right) = i\cosh u.$

66. $\cosh\left(\frac{i\pi}{2} \pm u\right) = \pm i \sinh u.$

67. $\tanh(u + i\pi) = \tanh u.$

C. — Inverse Hyperbolic Functions

68. $\sinh^{-1} u = \log\left(u + \sqrt{u^2+1}\right) = \cosh^{-1}\sqrt{u^2+1} = \int \frac{du}{(u^2+1)^{\frac{1}{2}}}.$

69. $\cosh^{-1} u = \log\left(u + \sqrt{u^2-1}\right) = \sinh^{-1}\sqrt{u^2-1} = \int \frac{du}{(u^2-1)^{\frac{1}{2}}}.$

70. $\tanh^{-1} u = \frac{1}{2}\log(1+u) - \frac{1}{2}\log(1-u) = \int \frac{du}{1-u^2}.$

71. $\coth^{-1} u = \frac{1}{2}\log(1+u) - \frac{1}{2}\log(u-1) = \int \frac{du}{1-u^2} = \tanh^{-1}\frac{1}{u}.$

72. $\operatorname{sech}^{-1} u = \log\left(\frac{1}{u} + \sqrt{\frac{1}{u^2} - 1}\right) = -\int \frac{du}{u(1-u^2)^{\frac{1}{2}}} = \cosh^{-1}\frac{1}{u}.$

73. $\operatorname{csch}^{-1} u = \log\left(\frac{1}{u} + \sqrt{\frac{1}{u^2} + 1}\right) = -\int \frac{du}{u(u^2+1)^{\frac{1}{2}}} = \sinh^{-1}\frac{1}{u}.$

74. $\sin^{-1} u = -i\sinh^{-1} iu = -i\log\left(iu + \sqrt{1-u^2}\right).$

75. $\cos^{-1} u = -i\cosh^{-1} u = -i\log\left(u + i\sqrt{1-u^2}\right).$

76. $\tan^{-1} u = -i\tanh^{-1} iu = \frac{1}{2i}\log(1+iu) - \frac{1}{2i}\log(1-iu).$

77. $\cot^{-1} u = i\coth^{-1} iu = \frac{1}{2i}\log(iu-1) - \frac{1}{2i}\log(iu+1).$

78. $\sin^{-1} iu = i\sinh^{-1} u = i\log\left(u + \sqrt{1+u^2}\right).$

79. $\cos^{-1} iu = -i\cosh^{-1} iu = \frac{\pi}{2} - i\log\left(u + \sqrt{1+u^2}\right).$

80. $\tan^{-1} iu = i\tanh^{-1} u = \frac{i}{2}\log(1+u) - \frac{i}{2}\log(1-u).$

81. $\cot^{-1} iu = -i\coth^{-1} u = -\frac{i}{2}\log(u+1) + \frac{i}{2}\log(u-1).$

82. $\cosh^{-1}\frac{1}{2}\left(u+\frac{1}{u}\right) = \sinh^{-1}\frac{1}{2}\left(u-\frac{1}{u}\right) = \tanh^{-1}\frac{u^2-1}{u^2+1}$

$$= 2\tanh^{-1}\frac{u-1}{u+1} = \log u.$$

83. $\tanh^{-1}\tan u = \frac{1}{2}\, gd\, 2u.$

84. $\tan^{-1}\tanh u = \frac{1}{2}\, gd^{-1}\, 2u.$

85. $\cosh^{-1}\csc 2u = -\sinh^{-1}\cot 2u = -\tanh^{-1}\cos 2u = \log\tan u.$

86. $\tanh^{-1}\tan^2\left(\frac{1}{4}\pi + \frac{1}{2}u\right) = \frac{1}{2}\log\csc u.$

87. $\tanh^{-1}\tan^2\frac{1}{2}u = \frac{1}{2}\log\sec u.$

88. $\cosh^{-1} u \pm \cosh^{-1} v = \cosh^{-1}\left[uv \pm \sqrt{(u^2-1)(v^2-1)}\right].$

89. $\sinh^{-1} u \pm \sinh^{-1} v = \sinh^{-1}\left[u\sqrt{1+v^2} \pm v\sqrt{1+u^2}\right].$

D. — Series

90. $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots.$ $(u^2 < \infty.)$

91. $\log u = (u-1) - \frac{1}{2}(u-1)^2 + \frac{1}{3}(u-1)^3 - \cdots.$ $(2 > u > 0.)$

92. $\log u = \frac{u-1}{u} + \frac{1}{2}\left(\frac{u-1}{u}\right)^2 + \frac{1}{3}\left(\frac{u-1}{u}\right)^3 + \cdots.$ $(u > \frac{1}{2}.)$

93. $\log u = 2\left[\frac{u-1}{u+1} + \frac{1}{3}\left(\frac{u-1}{u+1}\right)^3 + \frac{1}{5}\left(\frac{u-1}{u+1}\right)^5 + \cdots\right].$ $(u > 0.)$

94. $\log(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \cdots.$ $(u^2 < 1.)$

95. $\log\left(\frac{1+u}{1-u}\right) = 2\left[u + \frac{1}{3}u^3 + \frac{1}{5}u^5 + \frac{1}{7}u^7 + \cdots\right].$ $(u^2 < 1.)$

96. $\log\left(\frac{u+1}{u-1}\right) = 2\left[\frac{1}{u} + \frac{1}{3}\left(\frac{1}{u}\right)^3 + \frac{1}{5}\left(\frac{1}{u}\right)^5 + \cdots\right].$ $(u^2 > 1.)$

97. $\sinh u = u + \frac{u^3}{3!} + \frac{u^5}{5!} + \frac{u^7}{7!} + \cdots$ $(u^2 < \infty.)$

$$= u\left(1+\frac{u^2}{\pi^2}\right)\left(1+\frac{u^2}{2^2\pi^2}\right)\left(1+\frac{u^2}{3^2\pi^2}\right)\cdots. \quad (u^2 < \infty.)$$

98. $\cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + \cdots$ $(u^2 < \infty.)$

$$= \left(1 + \frac{4\,u^2}{\pi^2}\right)\left(1 + \frac{4\,u^2}{3^2\,\pi^2}\right)\left(1 + \frac{4\,u^2}{5^2\,\pi^2}\right) \cdots \quad (u^2 < \infty.)$$

99. $\tanh u = u - \frac{1}{3}u^3 + \frac{2}{15}u^5 - \frac{17}{315}u^7 + \cdots. \quad (u^2 < \frac{1}{4}\pi^2.)$

100. $u \coth u = 1 + \frac{1}{3}u^2 - \frac{1}{45}u^4 + \frac{2}{945}u^6 - \cdots. \quad (u^2 < \pi^2.)$

101. $\operatorname{sech} u = 1 - \frac{1}{2}u^2 + \frac{5}{24}u^4 - \frac{61}{720}u^6 + \cdots. \quad (u^2 < \frac{1}{4}\pi^2.)$

102. $u \operatorname{csch} u = 1 - \frac{1}{6}u^2 + \frac{7}{360}u^4 - \frac{31}{15{,}120}u^6 + \cdots. \quad (u^2 < \pi^2.)$

103. $gd\, u = \phi = u - \frac{1}{6}u^3 + \frac{1}{24}u^5 - \frac{61}{5040}u^7 + \cdots \quad (u \text{ small.})$

$$= \frac{\pi}{2} - \operatorname{sech} u - \frac{1}{2}\frac{\operatorname{sech}^3 u}{3} - \frac{1}{2}\frac{3}{4}\frac{\operatorname{sech}^5 u}{5} - \cdots. \quad (u \text{ large.})$$

104. $u = gd^{-1}\phi = \phi + \frac{1}{6}\phi^3 + \frac{1}{24}\phi^5 + \frac{61}{5040}\phi^7 + \cdots. \quad \left(\phi < \frac{\pi}{2}.\right)$

105. $\sinh^{-1} u = u - \frac{1}{2}\frac{u^3}{3} + \frac{1}{2}\frac{3}{4}\frac{u^5}{5} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{u^7}{7} + \cdots \quad (u^2 < 1.)$

$$= \log 2u + \frac{1}{2}\frac{1}{2u^2} - \frac{1}{2}\frac{3}{4}\frac{1}{4u^4} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{6u^6} - \cdots. \quad (u^2 > 1.)$$

106. $\cosh^{-1} u = \log 2u - \frac{1}{2}\frac{1}{2u^2} - \frac{1}{2}\frac{3}{4}\frac{1}{4u^4} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{6u^6} - \cdots. \quad (u^2 > 1.)$

107. $\tanh^{-1} u = u + \frac{1}{3}u^3 + \frac{1}{5}u^5 + \frac{1}{7}u^7 + \cdots. \quad (u^2 < 1.)$

108. $\coth^{-1} u = \tanh^{-1}\frac{1}{u} = \frac{1}{u} + \frac{1}{3u^3} + \frac{1}{5u^5} + \frac{1}{7u^7} + \cdots. \quad (u^2 > 1.)$

109. $\operatorname{sech}^{-1} u = \cosh^{-1}\frac{1}{u} = \log\frac{2}{u} - \frac{1}{2}\frac{u^2}{2} - \frac{1}{2}\frac{3}{4}\frac{u^4}{4} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{u^6}{6} - \cdots.$
$(u^2 < 1.)$

110. $\operatorname{csch}^{-1} u = \sinh^{-1}\frac{1}{u} = \frac{1}{u} - \frac{1}{2}\frac{1}{3u^3} + \frac{1}{2}\frac{3}{4}\frac{1}{5u^5} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{7u^7} + \cdots$
$(u^2 > 1.)$

$$= \log\frac{2}{u} + \frac{1}{2}\frac{u^2}{2} - \frac{1}{2}\frac{3}{4}\frac{u^4}{4} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{u^6}{6} - \cdots. \quad (u^2 < 1.)$$

E. — Derivatives

111. $\dfrac{de^u}{du} = e^u.$

112. $d\,\dfrac{\log_e u}{du} = \dfrac{1}{u}.$

113. $\dfrac{da^v}{du} = a^v \cdot \dfrac{dv}{du} \cdot \log_e a.$

114. $\dfrac{du^u}{du} = u^u(1 + \log_e u).$

115. $\dfrac{d \sinh u}{du} = \cosh u.$

116. $\dfrac{d \cosh u}{du} = \sinh u.$

117. $\dfrac{d \tanh u}{du} = \operatorname{sech}^2 u.$

118. $\dfrac{d \coth u}{du} = -\operatorname{csch}^2 u.$

119. $\dfrac{d \operatorname{sech} u}{du} = -\operatorname{sech} u.\ \tanh u.$

120. $\dfrac{d \operatorname{csch} u}{du} = -\operatorname{csch} u.\ \coth u.$

121. $\dfrac{d \sinh^{-1} u}{du} = \dfrac{1}{\sqrt{u^2+1}}.$

122. $\dfrac{d \cosh^{-1} u}{du} = \dfrac{1}{\sqrt{u^2-1}}.$

123. $\dfrac{d \tanh^{-1} u}{du} = \dfrac{1}{1-u^2}.$

124. $\dfrac{d \coth^{-1} u}{du} = \dfrac{1}{1-u^2}.$

125. $\dfrac{d \operatorname{sech}^{-1} u}{du} = \dfrac{-1}{u\sqrt{1-u^2}}.$

126. $\dfrac{d \operatorname{csch}^{-1} u}{du} = \dfrac{-1}{u\sqrt{u^2+1}}.$

127. $\dfrac{d \operatorname{gd} u}{du} = \operatorname{sech} u.$

128. $\dfrac{d \operatorname{gd}^{-1} u}{du} = \sec u.$

F. — Integrals. (Integration constants are omitted.)

129. $\int \sinh u\, du = \cosh u.$

130. $\int \cosh u\, du = \sinh u.$

131. $\int \tanh u\, du = \log \cosh u.$

132. $\int \coth u\, du = \log \sinh u.$

133. $\int \operatorname{sech} u\, du = 2 \tan^{-1} e^u = \operatorname{gd} u.$

134. $\int \operatorname{csch} u\, du = \log \tanh \dfrac{u}{2}.$

135. $\int \sinh^n u\, du = \dfrac{1}{n} \sinh^{n-1} u.\ \cosh u - \dfrac{n-1}{n} \int \sinh^{n-2} u\, du$

$= \dfrac{1}{n+1} \sinh^{n+1} u \cosh u - \dfrac{n+2}{n+1} \int \sinh^{n+2} u\, du.$

136. $\int \cosh^n u\, du = \dfrac{1}{n} \sinh u.\ \cosh^{n-1} u + \dfrac{n-1}{n} \int \cosh^{n-2} u\, du$

$= -\dfrac{1}{n+1} \sinh u \cosh^{n+1} u + \dfrac{n+2}{n+1} \int \cosh^{n+2} u\, du.$

137. $\int u \sinh u\, du = u \cosh u - \sinh u.$

138. $\int u \cosh u\, du = u \sinh u - \cosh u.$

139. $\int u^2 \sinh u\,du = (u^2 + 2)\cosh u - 2u\sinh u.$

140. $\int u^n \sinh u\,du = u^n \cosh u - nu^{n-1}\sinh u$
$$+ n(n-1)\int u^{n-2}\sinh u\,du.$$

141. $\int \sinh^2 u\,du = \frac{1}{2}(\sinh u \cosh u - u).$

142. $\int \sinh u.\ \cosh u\,du = \frac{1}{4}\cosh(2u).$

143. $\int \cosh^2 u\,du = \frac{1}{2}(\sinh u \cosh u + u).$

144. $\int \tanh^2 u\,du = u - \tanh u.$

145. $\int \coth^2 u\,du = u - \coth u.$

146. $\int \operatorname{sech}^2 u\,du = \tanh u.$

147. $\int \operatorname{sech}^3 u\,du = \frac{1}{2}\operatorname{sech} u \tanh u + \frac{1}{2}\operatorname{gd} u.$

148. $\int \operatorname{csch}^2 u\,du = -\coth u.$

149. $\int \sinh^{-1} u\,du = u\sinh^{-1} u - (1 + u^2)^{\frac{1}{2}}.$

150. $\int \cosh^{-1} u\,du = u\cosh^{-1} u - (u^2 - 1)^{\frac{1}{2}}.$

151. $\int \tanh^{-1} u\,du = u\tanh^{-1} u + \frac{1}{2}\log(1 - u^2).$

152. $\int u\sinh^{-1} u\,du = \frac{1}{4}\left[(2u^2 + 1)\sinh^{-1} u - u(1 + u^2)^{\frac{1}{2}}\right].$

153. $\int u\cosh^{-1} u\,du = \frac{1}{4}\left[(2u^2 - 1)\cosh^{-1} u - u(u^2 - 1)^{\frac{1}{2}}\right].$

154. $\int (\cosh a + \cosh u)^{-1}\,du = 2\operatorname{csch} a.\ \tanh^{-1}(\tanh\frac{1}{2}u.\ \tanh\frac{1}{2}a)$
$$= \operatorname{csch} a\left[\log\cosh\tfrac{1}{2}(u + a) - \log\cosh\tfrac{1}{2}(u - a)\right].$$

155. $\int (\cos a + \cosh u)^{-1}\,du = 2\csc a.\ \tan^{-1}(\tanh\frac{1}{2}u.\ \tan\frac{1}{2}a).$

156. $\int (1 + \cos a.\ \cosh u)^{-1}\,du = 2\csc a.\ \tanh^{-1}(\tanh\frac{1}{2}u.\ \tan\frac{1}{2}a).$

157. $\int \sinh u \cos u \, du = \frac{1}{2} (\cosh u. \ \cos u + \sinh u. \ \sin u).$

158. $\int \cosh u. \ \cos u \, du = \frac{1}{2} (\sinh u. \ \cos u + \cosh u. \ \sin u).$

159. $\int \sinh u. \ \sin u \, du = \frac{1}{2} (\cosh u. \ \sin u - \sinh u. \ \cos u).$

160. $\int \cosh u. \ \sin u \, du = \frac{1}{2} (\sinh u. \ \sin u - \cosh u. \ \cos u).$

161. $\int \sinh (mu) \sinh (nu) \, du$

$$= \frac{1}{m^2 - n^2} [m \sinh (nu) \cosh (mu) - n \cosh (nu) \sinh (mu)].$$

162. $\int \cosh (mu) \sinh (nu) \, du$

$$= \frac{1}{m^2 - n^2} [m \sinh (nu) \sinh (mu) - n \cosh (nu) \cosh (mu)].$$

163. $\int \cosh (mu) \cosh (nu) \, du$

$$= \frac{1}{m^2 - n^2} [m \sinh (mu) \cosh (nu) - n \sinh (nu) \cosh (mu)'.$$

164. $\int \sinh u \tanh u \, du = \sinh u - gd \, u.$

165. $\int \cosh u \coth u \, du = \cosh u + \log \tanh \frac{u}{2}.$

166. $\int \sec u \, du = \mathrm{gd}^{-1} u.$

167. $\int \sec^3 \phi \, d\phi = \int (1 + \tan^2 \phi)^{\frac{1}{2}} d \tan \phi = \frac{1}{2} \sec \phi \tan \phi + \frac{1}{2} \mathrm{gd}^{-1} \phi$

$$= \frac{1}{2} \tan \phi (1 + \tan^2 \phi)^{\frac{1}{2}} + \frac{1}{2} \sinh^{-1} (\tan \phi). \quad \text{Here } \phi = \mathrm{gd}\, u.$$

168. $\int \frac{du}{(u^2 + a^2)^{\frac{1}{2}}} = \sinh^{-1} \frac{u}{a}.$ $\qquad \int \frac{du}{(a^2 - u^2)^{\frac{1}{2}}} = \sin^{-1} \frac{u}{a}.$

169. $\int \frac{du}{(u^2 - a^2)^{\frac{1}{2}}} = \cosh^{-1} \frac{u}{a}.$ $\qquad \int \frac{-du}{(a^2 - u^2)^{\frac{1}{2}}} = \cos^{-1} \frac{u}{a}.$

170. $\int \frac{du}{(a^2 - u^2)_{u<a}} = \frac{1}{a} \tanh^{-1} \frac{u}{a}.$ $\qquad \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a}.$

171. $\int \frac{-du}{(u^2 - a^2)_{u>a}} = \frac{1}{a} \coth^{-1} \frac{u}{a}.$ $\qquad \int \frac{-du}{a^2 + u^2} = \frac{1}{a} \cot^{-1} \frac{u}{a}.$

172. $\int \frac{-du}{u (a^2 - u^2)^{\frac{1}{2}}} = \frac{1}{a} \mathrm{sech}^{-1} \frac{u}{a}.$ $\qquad \int \frac{du}{u (u^2 - a^2)^{\frac{1}{2}}} = \frac{1}{a} \sec^{-1} \frac{u}{a}.$

173. $\displaystyle\int \frac{-du}{u\,(a^2+u^2)^{\frac{1}{2}}} = \frac{1}{a}\operatorname{csch}^{-1}\frac{u}{a}.\quad \int \frac{-du}{u\,(u^2-a^2)^{\frac{1}{2}}} = \frac{1}{a}\csc^{-1}\frac{u}{a}.$

174. $$\int \frac{du}{(au^2+2\,bu+c)^{\frac{1}{2}}} = \frac{1}{\sqrt{a}}\sinh^{-1}\frac{au+b}{(ac-b^2)^{\frac{1}{2}}} \quad (a \text{ positive}, ac>b^2.)$$
$$= \frac{1}{\sqrt{a}}\cosh^{-1}\frac{au+b}{(b^2-ac)^{\frac{1}{2}}} \quad (a \text{ positive}, ac<b^2.)$$
$$= \frac{1}{\sqrt{-a}}\cos^{-1}\frac{au+b}{(b^2-ac)^{\frac{1}{2}}}. \quad (a \text{ negative.})$$

175. $$\int \frac{du}{(au^2+2\,bu+c)} = \frac{1}{(ac-b^2)^{\frac{1}{2}}}\tan^{-1}\frac{au+b}{(ac-b^2)^{\frac{1}{2}}} \quad (ac>b^2.)$$
$$= \frac{-1}{(b^2-ac)^{\frac{1}{2}}}\tanh^{-1}\frac{au+b}{(b^2-ac)^{\frac{1}{2}}} \quad \left(\begin{array}{c} ac<b^2. \\ au+b<(b^2-ac)^{\frac{1}{2}}.\end{array}\right)$$
$$= \frac{-1}{(b^2-ac)^{\frac{1}{2}}}\coth^{-1}\frac{au+b}{(b^2-ac)^{\frac{1}{2}}}. \quad \left(\begin{array}{c} ac<b^2. \\ au+b>(b^2-ac)^{\frac{1}{2}}.\end{array}\right)$$

176. $$\int \frac{du}{(a-u)\,(u-b)^{\frac{1}{2}}} = \frac{2}{(a-b)^{\frac{1}{2}}}\tanh^{-1}\sqrt{\frac{u-b}{a-b}},$$
or $$\frac{-2}{(b-a)^{\frac{1}{2}}}\tan^{-1}\sqrt{\frac{u-b}{b-a}},$$
or $$\frac{2}{(a-b)^{\frac{1}{2}}}\coth^{-1}\sqrt{\frac{u-b}{a-b}}. \quad \text{(The real form is to be taken.)}$$

177. $$\int \frac{du}{(a-u)\,(b-u)^{\frac{1}{2}}} = \frac{2}{(b-a)^{\frac{1}{2}}}\tanh^{-1}\sqrt{\frac{b-u}{b-a}},$$
or $$\frac{2}{(b-a)^{\frac{1}{2}}}\coth^{-1}\sqrt{\frac{b-u}{b-a}},$$
or $$\frac{-2}{(a-b)^{\frac{1}{2}}}\tan^{-1}\sqrt{\frac{b-u}{a-b}}. \quad \text{(The real form is to be taken.)}$$

178. $\displaystyle\int (u^2-a^2)^{\frac{1}{2}}\,du = \frac{1}{2}\,u\,(u^2-a^2)^{\frac{1}{2}} - \frac{1}{2}\,a^2\cosh^{-1}\frac{u}{a}.$

179. $\displaystyle\int (a^2-u^2)^{\frac{1}{2}}\,du = \frac{1}{2}\,u\,(a^2-u^2)^{\frac{1}{2}} + \frac{1}{2}\,a^2\sin^{-1}\frac{u}{a}.$

180. $\displaystyle\int (u^2+a^2)^{\frac{1}{2}}\,du = \frac{1}{2}\,u\,(u^2+a^2)^{\frac{1}{2}} + \frac{1}{2}\,a^2\sinh^{-1}\frac{u}{a}.$

181. $\displaystyle\int e^{au}\,du = \frac{e^{au}}{a}.$

182. $\displaystyle ue^{au}\,du = \frac{e^{au}}{a^2}\,(au-1).$

183. $\int u^m e^{au}\,du = \frac{u^m e^{au}}{a} - \frac{m}{a}\int u^{m-1}e^{au}\,du.$

184. $\int \frac{e^{au}\,du}{u^m} = \frac{1}{m-1}\left[-\frac{e^{au}}{u^{m-1}} + a\int\frac{e^{au}\,du}{u^{m-1}}\right].$

185. $\int a^{bu}\,du = \frac{a^{bu}}{b\log a}.$

186. $\int u^n a^u\,du = \frac{a^u u^n}{\log a} - \frac{na^u u^{n-1}}{(\log a)^2} + \frac{n(n-1)\,a^u u^{n-2}}{(\log a)^3} \cdots$

$\pm \frac{n(n-1)(n-2)\,\ldots\,2\cdot 1\,a^u}{(\log a)^{n+1}}.$

187. $\int\frac{a^u\,du}{u^n} = \frac{a^u}{n-1}\left[-\frac{1}{u^{n-1}} - \frac{\log a}{(n-2)\,u^{n-2}} - \frac{(\log a)^2}{(n-2)(n-3)\,u^{n-3}}\right.$

$\left. - \cdots + \frac{(\log a)^{n-1}}{(n-2)(n-3)\,\ldots\,2\cdot 1}\int\frac{a^u\,du}{u}\right].$

188. $\int\frac{a^u\,du}{u} = \log u + u\log a + \frac{(u\log a)^2}{2\cdot 2!} + \frac{(u\log a)^3}{3\cdot 3!} + \cdots.$

189. $\int\frac{du}{1+e^u} = \log\frac{e^u}{1+e^u}.$

190. $\int\frac{du}{a+be^{mu}} = \frac{1}{am}\,[mu - \log(a+be^{mu})].$

191. $\int\frac{du}{ae^{mu}+be^{-mu}} = \frac{1}{m(ab)^{\frac{1}{2}}}\tan^{-1}\left(e^{mu}\sqrt{\frac{a}{b}}\right).$

192. $\int\frac{du}{(a+be^{mu})^{\frac{1}{2}}} = \frac{1}{m\sqrt{a}}\left[\log\left(\sqrt{a+be^{mu}} - \sqrt{a}\right)\right.$

$\left. - \log\left(\sqrt{a+be^{mu}} + \sqrt{a}\right)\right].$

193. $\int\frac{ue^u\,du}{(1+u)^2} = \frac{e^u}{1+u}.$

194. $\int e^{au}\log u\,du = \frac{e^{au}\log u}{a} - \frac{1}{a}\int\frac{e^{au}\,du}{u}.$

195. $\int\log u\,du = u\log u - u.$

196. $\int u^m\log u\,du = u^{m+1}\left[\frac{\log u}{m+1} - \frac{1}{(m+1)^2}\right].$

197. $\int(\log u)^n\,du = u(\log u)^n - n\int(\log u)^{n-1}\,du.$

198. $\int u^m(\log u)^n\,du = \frac{u^{m+1}(\log u)^n}{m+1} - \frac{n}{m+1}\int u^m(\log u)^{n-1}\,du.$

199. $\int\frac{(\log u)^n\,du}{u} = \frac{(\log u)^{n+1}}{n+1}.$

200. $$\int \frac{du}{\log u} = \log(\log u) + \log u + \frac{(\log u)^2}{2 \cdot 2!} + \frac{(\log u)^3}{3 \cdot 3!} + \cdots.$$

201. $$\int \frac{du}{(\log u)^n} = -\frac{u}{(n-1)(\log u)^{n-1}} + \frac{1}{n-1}\int \frac{du}{(\log u)^{n-1}}.$$

202. $$\int \frac{u^m\,du}{(\log u)^n} = -\frac{u^{m+1}}{(n-1)(\log u)^{n-1}} + \frac{m+1}{n-1}\int \frac{u^m\,du}{(\log u)^{n-1}}.$$

203. $$\int \frac{u^m\,du}{\log u} = \int \frac{e^{-y}}{y}\,dy, \quad \text{where} \quad y = -(m+1)\log u.$$

204. $$\int \frac{du}{u \log u} = \log(\log u).$$

205. $$\int \frac{du}{u(\log u)^n} = -\frac{1}{(n-1)(\log u)^{n-1}}.$$

206. $$\int (a+bu)^m \log u\,du = \frac{1}{b(m+1)}\left[(a+bu)^{m+1}\log u - \int \frac{(a+bu)^{m+1}\,du}{u}\right].$$

207. $$\int u^m \log(a+bu)\,du = \frac{1}{m+1}\left[u^{m+1}\log(a+bu) - b\int \frac{u^{m+1}\,du}{a+bu}\right].$$

208. $$\int \frac{\log(a+bu)\,du}{u} = \log a \cdot \log u + \frac{bu}{a} - \frac{1}{2^2}\left(\frac{bu}{a}\right)^2 + \frac{1}{3^2}\left(\frac{bu}{a}\right)^3 - \cdots$$
$$= \frac{1}{2}(\log bu)^2 - \frac{a}{bu} + \frac{1}{2^2}\left(\frac{a}{bu}\right)^2 - \frac{1}{3^2}\left(\frac{a}{bu}\right)^3 + \cdots.$$

209. $$\int \frac{\log u\,du}{(a+bu)^m} = \frac{1}{b(m-1)}\left[-\frac{\log u}{(a+bu)^{m-1}} + \int \frac{du}{u(a+bu)^{m-1}}\right].$$

210. $$\int \frac{\log u\,du}{a+bu} = \frac{1}{b}\log u \cdot \log(a+bu) - \frac{1}{b}\int \frac{\log(a+bu)}{u}\,du.$$

211. $$\int (a+bu)\log u\,du = \frac{(a+bu)^2}{2b}\log u - \frac{a^2\log u}{2b} - au - \frac{1}{4}bu^2.$$

212. $$\int \frac{\log u\,du}{(a+bu)^{\frac{1}{2}}} =$$
$$\frac{2}{b}\Big[(\log u - 2)\sqrt{(a+bu)} + \sqrt{a}\log\left(\sqrt{a+bu} + \sqrt{a}\right) - \sqrt{a}\log\left(\sqrt{a+bu} - \sqrt{a}\right)\Big], \text{ if } a > 0,$$
$$= \frac{2}{b}\left[(\log u - 2)\sqrt{(a+bu)} + 2\sqrt{-a}\tan^{-1}\sqrt{\frac{a+bu}{-a}}\right], \text{ if } a < 0.$$

213. $\displaystyle\int_0^\infty e^{-a^2u^2}\,du = \frac{\sqrt{\pi}}{2a} = \frac{1}{2a}\,\Gamma\left(\frac{1}{2}\right).$

214. $\displaystyle\int_0^\infty u^n e^{-au}\,du = \Gamma\,\frac{(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}}.$

215. $\displaystyle\int_0^\infty u^{2n} e^{-au^2}\,du = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}\,a^n}\sqrt{\frac{\pi}{a}}.$

216. $\displaystyle\int_0^\infty e^{-u^2-\frac{a^2}{u^2}}\,du = \frac{e^{-2a}}{2}\sqrt{\frac{\pi}{a}}.$ $(a > 0.)$

217. $\displaystyle\int_0^\infty e^{-nu}\sqrt{u}\,du = \frac{1}{2n}\sqrt{\frac{\pi}{n}}.$

218. $\displaystyle\int_0^\infty \frac{e^{-nu}}{\sqrt{u}}\,du = \sqrt{\frac{\pi}{n}}.$ $(n > 0.)$

219. $\displaystyle\int_0^\infty \frac{du}{\sinh(nu)} = \frac{\pi}{2n}.$

220. $\displaystyle\int_0^\infty \frac{u\,du}{\sinh(nu)} = \frac{\pi^2}{4n^2}.$

221. $\displaystyle\int_0^{i\pi} \sinh(mu)\cdot\sinh(nu)\,du = \int_0^{i\pi}\cosh(mu)\cdot\cosh(nu)\,du$
$= 0$, if m is different from n.

222. $\displaystyle\int_0^{i\pi} \cosh^2(mu)\,du = -\int_0^{i\pi}\sinh^2(mu)\,du = \frac{i\pi}{2}.$

223. $\displaystyle\int_{-i\pi}^{+i\pi} \sinh(mu)\,du = 0.$

224. $\displaystyle\int_0^{i\pi} \cosh(mu)\,du = 0.$

225. $\displaystyle\int_{-i\pi}^{i\pi} \sinh(mu)\cosh(nu)\,du = 0.$

226. $\displaystyle\int_0^{i\pi} \sinh(mu)\cosh(mu)\,du = 0.$

227. $\displaystyle\int_0^1 \frac{\log u}{1-u}\,du = -\frac{\pi^2}{6}.$

228. $\displaystyle\int_0^1 \frac{\log u}{1+u}\,du = -\frac{\pi^2}{12}.$

229. $\displaystyle\int_0^1 \frac{\log u}{1-u^2}\,du = -\frac{\pi^2}{8}.$

230. $\displaystyle\int_0^1 \log\left(\frac{1+u}{1-u}\right)\cdot\frac{du}{u} = \frac{\pi^2}{4}.$

231. $\displaystyle\int_0^1 \frac{\log u\,du}{(1-u^2)^{\frac{1}{2}}} = -\frac{\pi}{2}\log 2.$

232. $\displaystyle\int_0^1 \frac{(u^p - u^q)\,du}{\log u} = \log\frac{p+1}{q+1}$, if $p + 1 > 0, q + 1 > 0$.

233. $\displaystyle\int_0^1 (\log u)^n\,du = (-1)^n \cdot n!$.

234. $\displaystyle\int_0^1 \left(\log\frac{1}{u}\right)^{\frac{1}{2}} du = \frac{1}{2}\sqrt{\pi}$.

235. $\displaystyle\int_0^1 \left(\log\frac{1}{u}\right)^n du = n!$.

236. $\displaystyle\int_0^1 \frac{du}{\left(\log\frac{1}{u}\right)^{\frac{1}{2}}} = \sqrt{\pi}$.

237. $\displaystyle\int_0^1 u^m \log\left(\frac{1}{u}\right)^n du = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$, if $m + 1 > 0, n + 1 > 0$.

238. $\displaystyle\int_0^\infty \log\left(\frac{e^u + 1}{e^u - 1}\right) du = \frac{\pi^2}{4}$.

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS

x	log cosh x	log sinh x	log tanh x	$\log \frac{\sinh x}{x}$	$\log \frac{x}{\tanh x}$
0.000	0.000 000	−Inf.	−Inf.	0.000 000	0.000 000
0.001	0.000 000	7.000 000	7.000 000	0.000 000	0.000 000
0.002	0.000 001	7.301 030	7.301 029	0.000 000	0.000 001
0.003	0.000 002	7.477 122	7.477 120	0.000 001	0.000 001
0.004	0.000 003	7.602 061	7.602 058	0.000 001	0.000 002
0.005	0.000 005	7.698 972	7.698 966	0.000 002	0.000 004
0.006	0.000 008	7.778 154	7.778 146	0.000 003	0.000 005
0.007	0.000 011	7.845 102	7.845 091	0.000 004	0.000 007
0.008	0.000 014	7.903 095	7.903 081	0.000 005	0.000 009
0.009	0.000 018	7.954 248	7.954 231	0.000 006	0.000 012
0.010	0.000 022	8.000 007	7.999 986	0.000 007	0.000 014
0.011	0.000 026	8.041 401	8.041 375	0.000 009	0.000 018
0.012	0.000 031	8.079 192	8.079 160	0.000 010	0.000 021
0.013	0.000 037	8.113 956	8.113 919	0.000 012	0.000 024
0.014	0.000 043	8.146 142	8.146 100	0.000 014	0.000 028
0.015	0.000 049	8.176 108	8.176 059	0.000 016	0.000 033
0.016	0.000 056	8.204 139	8.204 083	0.000 019	0.000 037
0.017	0.000 063	8.230 470	8.230 407	0.000 021	0.000 042
0.018	0.000 070	8.255 296	8.255 226	0.000 023	0.000 047
0.019	0.000 078	8.278 780	8.278 701	0.000 026	0.000 052
0.020	0.000 087	8.301 059	8.300 972	0.000 029	0.000 058
0.021	0.000 096	8.322 251	8.322 155	0.000 032	0.000 064
0.022	0.000 105	8.342 458	8.342 353	0.000 035	0.000 070
0.023	0.000 115	8.361 766	8.361 651	0.000 038	0.000 077
0.024	0.000 125	8.380 253	8.380 128	0.000 042	0.000 083
0.025	0.000 136	8.397 985	8.397 850	0.000 045	0.000 090
0.026	0.000 147	8.415 022	8.414 876	0.000 049	0.000 098
0.027	0.000 158	8.431 417	8.431 258	0.000 053	0.000 106
0.028	0.000 170	8.447 215	8.447 045	0.000 057	0.000 113
0.029	0.000 183	8.462 459	8.462 276	0.000 061	0.000 122
0.030	0.000 195	8.477 186	8.476 991	0.000 065	0.000 130
0.031	0.000 209	8.491 431	8.491 223	0.000 070	0.000 139
0.032	0.000 222	8.505 224	8.505 002	0.000 074	0.000 148
0.033	0.000 236	8.518 593	8.518 356	0.000 079	0.000 158
0.034	0.000 251	8.531 563	8.531 312	0.000 084	0.000 167
0.035	0.000 266	8.544 157	8.543 891	0.000 089	0.000 177
0.036	0.000 281	8.556 396	8.556 115	0.000 094	0.000 188
0.037	0.000 297	8.568 301	8.568 004	0.000 099	0.000 198
0.038	0.000 313	8.579 888	8.579 575	0.000 105	0.000 209
0.039	0.000 330	8.591 175	8.590 844	0.000 110	0.000 220
0.040	0.000 347	8.602 176	8.601 828	0.000 116	0.000 232
0.041	0.000 365	8.612 906	8.612 541	0.000 122	0.000 243
0.042	0.000 383	8.623 377	8.622 994	0.000 128	0.000 255
0.043	0.000 401	8.633 602	8.633 201	0.000 134	0.000 268
0.044	0.000 420	8.643 593	8.643 173	0.000 140	0.000 280
0.045	0.000 440	8.653 359	8.652 920	0.000 147	0.000 293
0.046	0.000 459	8.662 911	8.662 452	0.000 153	0.000 306
0.047	0.000 480	8.672 258	8.671 778	0.000 160	0.000 320
0.048	0.000 500	8.681 408	8.680 908	0.000 167	0.000 333
0.049	0.000 521	8.690 370	8.689 849	0.000 174	0.000 347

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS

(Continued)

x	log cosh x	log sinh x	log tanh x	$\log \frac{\sinh x}{x}$	$\log \frac{x}{\tanh x}$
0.050	0.000 543	8.699 151	8.698 608	0.000 181	0.000 362
0.051	0.000 565	8.707 758	8.707 194	0.000 188	0.000 376
0.052	0.000 587	8.716 199	8.715 612	0.000 196	0.000 391
0.053	0.000 610	8.724 479	8.723 869	0.000 203	0.000 406
0.054	0.000 633	8.732 605	8.731 972	0.000 211	0.000 422
0.055	0.000 657	8.740 582	8.739 925	0.000 219	0.000 438
0.056	0.000 681	8.748 415	8.747 734	0.000 227	0.000 454
0.057	0.000 705	8.756 110	8.755 405	0.000 235	0.000 470
0.058	0.000 730	8.763 671	8.762 941	0.000 243	0.000 487
0.059	0.000 755	8.771 104	8.770 348	0.000 252	0.000 504
0.060	0.000 781	8.778 412	8.777 631	0.000 261	0.000 521
0.061	0.000 808	8.785 599	8.784 792	0.000 269	0.000 538
0.062	0.000 834	8.792 670	8.791 836	0.000 278	0.000 556
0.063	0.000 861	8.799 628	8.798 767	0.000 287	0.000 574
0.064	0.000 889	8.806 476	8.805 588	0.000 296	0.000 592
0.065	0.000 917	8.813 219	8.812 302	0.000 306	0.000 611
0.066	0.000 945	8.819 859	8.818 914	0.000 315	0.000 630
0.067	0.000 974	8.826 400	8.825 426	0.000 325	0.000 649
0.068	0.001 003	8.832 844	8.831 840	0.000 335	0.000 669
0.069	0.001 033	8.839 194	8.838 161	0.000 345	0.000 688
0.070	0.001 063	8.845 453	8.844 390	0.000 355	0.000 709
0.071	0.001 094	8.851 623	8.850 529	0.000 365	0.000 729
0.072	0.001 125	8.857 708	8.856 583	0.000 375	0.000 750
0.073	0.001 156	8.863 709	8.862 552	0.000 386	0.000 770
0.074	0.001 188	8.869 628	8.868 440	0.000 396	0.000 792
0.075	0.001 220	8.875 468	8.874 248	0.000 407	0.000 813
0.076	0.001 253	8.881 232	8.879 979	0.000 418	0.000 835
0.077	0.001 286	8.886 920	8.885 634	0.000 429	0.000 857
0.078	0.001 320	8.892 535	8.891 215	0.000 440	0.000 880
0.079	0.001 354	8.898 079	8.896 725	0.000 452	0.000 902
0.080	0.001 388	8.903 553	8.902 165	0.000 463	0.000 925
0.081	0.001 423	8.908 960	8.907 537	0.000 475	0.000 948
0.082	0.001 458	8.914 300	8.912 842	0.000 487	0.000 972
0.083	0.001 494	8.919 577	8.918 082	0.000 499	0.000 996
0.084	0.001 530	8.924 790	8.923 260	0.000 511	0.001 020
0.085	0.001 567	8.929 942	8.928 375	0.000 523	0.001 044
0.086	0.001 604	8.935 034	8.933 430	0.000 535	0.001 069
0.087	0.001 642	8.940 067	8.938 425	0.000 548	0.001 094
0.088	0.001 679	8.945 043	8.943 364	0.000 560	0.001 119
0.089	0.001 718	8.949 963	8.948 245	0.000 573	0.001 145
0.090	0.001 757	8.954 829	8.953 072	0.000 586	0.001 170
0.091	0.001 796	8.959 641	8.957 845	0.000 599	0.001 196
0.092	0.001 835	8.964 400	8.962 565	0.000 612	0.001 223
0.093	0.001 875	8.969 109	8.967 233	0.000 626	0.001 250
0.094	0.001 916	8.973 767	8.971 851	0.000 639	0.001 277
0.095	0.001 957	8.978 377	8.976 420	0.000 653	0.001 304
0.096	0.001 998	8.982 938	8.980 940	0.000 667	0.001 331
0.097	0.002 040	8.987 453	8.985 413	0.000 681	0.001 359
0.098	0.002 082	8.991 921	8.989 839	0.000 695	0.001 387
0.099	0.002 125	8.996 344	8.994 220	0.000 709	0.001 416

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(*Continued*)

x	log cosh x	log sinh x	log tanh x	$\log \frac{\sinh x}{x}$	$\log \frac{x}{\tanh x}$
0.100	0.002 168	9.000 724	8.998 556	0.000 724	0.001 444
0.101	0.002 211	9.005 059	9.002 848	0.000 738	0.001 473
0.102	0.002 255	9.009 353	9.007 098	0.000 753	0.001 502
0.103	0.002 300	9.013 605	9.011 305	0.000 768	0.001 532
0.104	0.002 344	9.017 816	9.015 472	0.000 783	0.001 562
0.105	0.002 390	9.021 987	9.019 597	0.000 798	0.001 592
0.106	0.002 435	9.026 119	9.023 684	0.000 813	0.001 622
0.107	0.002 481	9.030 212	9.027 731	0.000 828	0.001 653
0.108	0.002 528	9.034 268	9.031 740	0.000 844	0.001 684
0.109	0.002 575	9.038 286	9.035 711	0.000 860	0.001 715
0.110	0.002 622	9.042 268	9.039 646	0.000 875	0.001 747
0.111	0.002 670	9.046 214	9.043 544	0.000 891	0.001 779
0.112	0.002 718	9.050 126	9.047 407	0.000 908	0.001 811
0.113	0.002 767	9.054 002	9.051 235	0.000 924	0.001 843
0.114	0.002 816	9.057 845	9.055 029	0.000 940	0.001 876
0.115	0.002 865	9.061 655	9.058 789	0.000 957	0.001 909
0.116	0.002 915	9.065 432	9.062 516	0.000 974	0.001 942
0.117	0.002 966	9.069 176	9.066 210	0.000 990	0.001 975
0.118	0.003 017	9.072 889	9.069 873	0.001 007	0.002 009
0.119	0.003 068	9.076 571	9.073 504	0.001 025	0.002 043
0.120	0.003 119	9.080 223	9.077 104	0.001 042	0.002 078
0.121	0.003 172	9.083 845	9.080 673	0.001 059	0.002 112
0.122	0.003 224	9.087 437	9.084 213	0.001 077	0.002 147
0.123	0.003 277	9.091 000	9.087 723	0.001 095	0.002 182
0.124	0.003 330	9.094 534	9.091 204	0.001 112	0.002 218
0.125	0.003 384	9.098 040	9.094 656	0.001 130	0.002 254
0.126	0.003 438	9.101 519	9.098 081	0.001 149	0.002 290
0.127	0.003 493	9.104 971	9.101 478	0.001 167	0.002 326
0.128	0.003 548	9.108 395	9.104 847	0.001 185	0.002 363
0.129	0.003 604	9.111 794	9.108 190	0.001 204	0.002 400
0.130	0.003 659	9.115 166	9.111 506	0.001 223	0.002 437
0.131	0.003 716	9.118 513	9.114 797	0.001 241	0.002 474
0.132	0.003 773	9.121 834	9.118 062	0.001 260	0.002 512
0.133	0.003 830	9.125 131	9.121 301	0.001 280	0.002 550
0.134	0.003 887	9.128 404	9.124 516	0.001 299	0.002 589
0.135	0.003 946	9.131 652	9.127 707	0.001 318	0.002 627
0.136	0.004 004	9.134 877	9.130 873	0.001 338	0.002 666
0.137	0.004 063	9.138 078	9.134 015	0.001 358	0.002 705
0.138	0.004 122	9.141 257	9.137 134	0.001 378	0.002 745
0.139	0.004 182	9.144 412	9.140 230	0.001 398	0.002 784
0.140	0.004 242	9.147 546	9.143 304	0.001 418	0.002 824
0.141	0.004 303	9.150 657	9.146 354	0.001 438	0.002 865
0.142	0.004 364	9.153 747	9.149 383	0.001 459	0.002 905
0.143	0.004 425	9.156 815	9.152 390	0.001 479	0.002 946
0.144	0.004 487	9.159 862	9.155 375	0.001 500	0.002 987
0.145	0.004 550	9.162 889	9.158 339	0.001 521	0.003 029
0.146	0.004 612	9.165 895	9.161 282	0.001 542	0.003 071
0.147	0.004 676	9.168 880	9.164 205	0.001 563	0.003 113
0.148	0.004 739	9.171 846	9.167 107	0.001 584	0.003 155
0.149	0.004 803	9.174 792	9.169 989	0.001 606	0.003 197

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	log sinh x	log tanh x	$\log \frac{\sinh x}{x}$	$\log \frac{x}{\tanh x}$
0.150	0.004 868	9.177 719	9.172 851	0.001 627	0.003 240
0.151	0.004 932	9.180 626	9.175 694	0.001 649	0.003 283
0.152	0.004 998	9.183 515	9.178 517	0.001 671	0.003 327
0.153	0.005 063	9.186 385	9.181 321	0.001 693	0.003 370
0.154	0.005 130	9.189 236	9.184 106	0.001 715	0.003 414
0.155	0.005 196	9.192 069	9.186 873	0.001 738	0.003 459
0.156	0.005 263	9.194 885	9.189 621	0.001 760	0.003 503
0.157	0.005 331	9.197 682	9.192 352	0.001 783	0.003 548
0.158	0.005 398	9.200 463	9.195 064	0.001 805	0.003 593
0.159	0.005 467	9.203 225	9.197 759	0.001 828	0.003 638
0.160	0.005 535	9.205 971	9.200 436	0.001 851	0.003 684
0.161	0.005 605	9.208 700	9.203 096	0.001 875	0.003 730
0.162	0.005 674	9.211 413	9.205 739	0.001 898	0.003 776
0.163	0.005 744	9.214 109	9.208 365	0.001 921	0.003 823
0.164	0.005 814	9.216 789	9.210 975	0.001 945	0.003 869
0.165	0.005 885	9.219 453	9.213 568	0.001 969	0.003 916
0.166	0.005 956	9.222 101	9.216 144	0.001 993	0.003 964
0.167	0.006 028	9.224 733	9.218 705	0.002 017	0.004 011
0.168	0.006 100	9.227 350	9.221 250	0.002 041	0.004 059
0.169	0.006 173	9.229 952	9.223 779	0.002 065	0.004 107
0.170	0.006 246	9.232 539	9.226 293	0.002 090	0.004 156
0.171	0.006 319	9.235 111	9.228 792	0.002 114	0.004 204
0.172	0.006 393	9.237 668	9.231 275	0.002 139	0.004 253
0.173	0.006 467	9.240 210	9.233 743	0.002 164	0.004 303
0.174	0.006 541	9.242 738	9.236 197	0.002 189	0.004 352
0.175	0.006 616	9.245 253	9.238 636	0.002 214	0.004 402
0.176	0.006 692	9.247 752	9.241 061	0.002 240	0.004 452
0.177	0.006 768	9.250 239	9.243 471	0.002 265	0.004 502
0.178	0.006 844	9.252 711	9.245 867	0.002 291	0.004 553
0.179	0.006 921	9.255 170	9.248 249	0.002 317	0.004 604
0.180	0.006 998	9.257 615	9.250 617	0.002 343	0.004 655
0.181	0.007 075	9.260 047	9.252 972	0.002 369	0.004 707
0.182	0.007 153	9.262 466	9.255 313	0.002 395	0.004 758
0.183	0.007 232	9.264 872	9.257 641	0.002 421	0.004 810
0.184	0.007 311	9.267 266	9.259 955	0.002 448	0.004 863
0.185	0.007 390	9.269 646	9.262 256	0.002 474	0.004 915
0.186	0.007 470	9.272 014	9.264 545	0.002 501	0.004 968
0.187	0.007 550	9.274 370	9.266 820	0.002 528	0.005 021
0.188	0.007 630	9.276 713	9.269 083	0.002 555	0.005 075
0.189	0.007 711	9.279 044	9.271 333	0.002 583	0.005 128
0.190	0.007 792	9.281 363	9.273 571	0.002 610	0.005 182
0.191	0.007 874	9.283 671	9.275 797	0.002 637	0.005 237
0.192	0.007 956	9.285 966	9.278 010	0.002 665	0.005 291
0.193	0.008 039	9.288 250	9.280 211	0.002 693	0.005 346
0.194	0.008 122	9.290 523	9.282 401	0.002 721	0.005 401
0.195	0.008 205	9.292 783	9.284 578	0.002 749	0.005 456
0.196	0.008 289	9.295 033	9.286 744	0.002 777	0.005 512
0.197	0.008 373	9.297 272	9.288 898	0.002 805	0.005 568
0.198	0.008 458	9.299 499	9.291 041	0.002 834	0.005 624
0.199	0.008 543	9.301 716	9.293 173	0.002 863	0.005 680

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	log sinh x	log tanh x	$\log \frac{\sinh x}{x}$	$\log \frac{x}{\tanh x}$
0.200	0.008 629	9.303 921	9.295 293	0.002 891	0.005 737
0.201	0.008 715	9.306 116	9.297 402	0.002 920	0.005 794
0.202	0.008 801	9.308 301	9.299 500	0.002 949	0.005 851
0.203	0.008 888	9.310 475	9.301 587	0.002 979	0.005 909
0.204	0.008 975	9.312 638	9.303 663	0.003 008	0.005 967
0.205	0.009 062	9.314 791	9.305 729	0.003 038	0.006 025
0.206	0.009 150	9.316 935	9.307 784	0.003 067	0.006 083
0.207	0.009 239	9.319 067	9.309 829	0.003 097	0.006 142
0.208	0.009 328	9.321 190	9.311 863	0.003 127	0.006 201
0.209	0.009 417	9.323 303	9.313 886	0.003 157	0.006 260
0.210	0.009 507	9.325 407	9.315 900	0.003 187	0.006 319
0.211	0.009 597	9.327 500	9.317 904	0.003 218	0.006 379
0.212	0.009 687	9.329 584	9.319 897	0.003 248	0.006 439
0.213	0.009 778	9.331 659	9.321 880	0.003 279	0.006 499
0.214	0.009 869	9.333 724	9.323 854	0.003 310	0.006 560
0.215	0.009 961	9.335 779	9.325 818	0.003 341	0.006 621
0.216	0.010 053	9.337 826	9.327 772	0.003 372	0.006 682
0.217	0.010 146	9.339 863	9.329 717	0.003 403	0.006 743
0.218	0.010 239	9.341 891	9.331 652	0.003 434	0.006 805
0.219	0.010 332	9.343 910	9.333 578	0.003 466	0.006 866
0.220	0.010 426	9.345 920	9.335 494	0.003 498	0.006 929
0.221	0.010 520	9.347 922	9.337 401	0.003 529	0.006 991
0.222	0.010 615	9.349 914	9.339 299	0.003 561	0.007 054
0.223	0.010 710	9.351 898	9.341 188	0.003 594	0.007 117
0.224	0.010 806	9.353 874	9.343 068	0.003 626	0.007 180
0.225	0.010 902	9.355 841	9.344 939	0.003 658	0.007 243
0.226	0.010 998	9.357 799	9.346 801	0.003 691	0.007 307
0.227	0.011 095	9.359 749	9.348 655	0.003 723	0.007 371
0.228	0.011 192	9.361 691	9.350 499	0.003 756	0.007 435
0.229	0.011 289	9.363 625	9.352 335	0.003 789	0.007 500
0.230	0.011 387	9.365 550	9.354 163	0.003 822	0.007 565
0.231	0.011 486	9.367 468	9.355 982	0.003 856	0.007 630
0.232	0.011 584	9.369 377	9.357 793	0.003 889	0.007 695
0.233	0.011 684	9.371 278	9.359 595	0.003 922	0.007 761
0.234	0.011 783	9.373 172	9.361 389	0.003 956	0.007 827
0.235	0.011 883	9.375 058	9.363 175	0.003 990	0.007 893
0.236	0.011 984	9.376 936	9.364 952	0.004 024	0.007 960
0.237	0.012 084	9.378 806	9.366 722	0.004 058	0.008 026
0.238	0.012 186	9.380 669	9.368 484	0.004 092	0.008 093
0.239	0.012 287	9.382 525	9.370 237	0.004 127	0.008 161
0.240	0.012 389	9.384 372	9.371 983	0.004 161	0.008 228
0.241	0.012 492	9.386 213	9.373 721	0.004 196	0.008 296
0.242	0.012 595	9.388 046	9.375 451	0.004 231	0.008 364
0.243	0.012 698	9.389 872	9.377 174	0.004 266	0.008 432
0.244	0.012 802	9.391 691	9.378 889	0.004 301	0.008 501
0.245	0.012 906	9.393 502	9.380 596	0.004 336	0.008 570
0.246	0.013 010	9.395 307	9.382 296	0.004 371	0.008 639
0.247	0.013 115	9.397 104	9.383 989	0.004 407	0.008 708
0.248	0.013 221	9.398 894	9.385 674	0.004 443	0.008 778
0.249	0.013 326	9.400 678	9.387 351	0.004 479	0.008 848

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(*Continued*)

x	log cosh x	log sinh x	log tanh x	$\log \frac{\sinh x}{x}$	$\log \frac{x}{\tanh x}$
0.250	0.013 433	9.402 455	9.389 022	0.004 515	0.008 918
0.251	0.013 539	9.404 224	9.390 685	0.004 551	0.008 989
0.252	0.013 646	9.405 987	9.392 341	0.004 587	0.009 059
0.253	0.013 754	9.407 744	9.393 990	0.004 623	0.009 130
0.254	0.013 861	9.409 494	9.395 632	0.004 660	0.009 202
0.255	0.013 970	9.411 237	9.397 267	0.004 697	0.009 273
0.256	0.014 078	9.412 973	9.398 895	0.004 733	0.009 345
0.257	0.014 187	9.414 703	9.400 516	0.004 770	0.009 417
0.258	0.014 297	9.416 427	9.402 130	0.004 807	0.009 489
0.259	0.014 406	9.418 144	9.403 738	0.004 845	0.009 562
0.260	0.014 517	9.419 855	9.405 339	0.004 882	0.009 635
0.261	0.014 627	9.421 560	9.406 933	0.004 920	0.009 708
0.262	0.014 738	9.423 259	9.408 520	0.004 957	0.009 781
0.263	0.014 850	9.424 951	9.410 101	0.004 995	0.009 855
0.264	0.014 962	9.426 637	9.411 675	0.005 033	0.009 929
0.265	0.015 074	9.428 317	9.413 243	0.005 071	0.010 003
0.266	0.015 187	9.429 991	9.414 804	0.005 109	0.010 077
0.267	0.015 300	9.431 659	9.416 359	0.005 148	0.010 152
0.268	0.015 413	9.433 321	9.417 908	0.005 186	0.010 227
0.269	0.015 527	9.434 977	9.419 450	0.005 225	0.010 302
0.270	0.015 641	9.436 628	9.420 986	0.005 264	0.010 377
0.271	0.015 756	9.438 272	9.422 516	0.005 303	0.010 453
0.272	0.015 871	9.439 911	9.424 040	0.005 342	0.010 529
0.273	0.015 987	9.441 544	9.425 557	0.005 381	0.010 605
0.274	0.016 103	9.443 171	9.427 069	0.005 421	0.010 682
0.275	0.016 219	9.444 793	9.428 574	0.005 460	0.010 759
0.276	0.016 336	9.446 409	9.430 073	0.005 500	0.010 836
0.277	0.016 453	9.448 019	9.431 567	0.005 540	0.010 913
0.278	0.016 570	9.449 624	9.433 054	0.005 580	0.010 991
0.279	0.016 688	9.451 224	9.434 536	0.005 620	0.011 068
0.280	0.016 806	9.452 818	9.436 012	0.005 660	0.011 146
0.281	0.016 925	9.454 407	9.437 482	0.005 700	0.011 225
0.282	0.017 044	9.455 990	9.438 946	0.005 741	0.011 303
0.283	0.017 164	9.457 568	9.440 404	0.005 782	0.011 382
0.284	0.017 284	9.459 141	9.441 857	0.005 822	0.011 461
0.285	0.017 404	9.460 708	9.443 304	0.005 863	0.011 541
0.286	0.017 525	9.462 271	9.444 746	0.005 905	0.011 620
0.287	0.017 646	9.463 828	9.446 182	0.005 946	0.011 700
0.288	0.017 767	9.465 380	9.447 612	0.005 987	0.011 780
0.289	0.017 889	9.466 927	9.449 037	0.006 029	0.011 861
0.290	0.018 012	9.468 468	9.450 457	0.006 070	0.011 941
0.291	0.018 134	9.470 005	9.451 871	0.006 112	0.012 022
0.292	0.018 258	9.471 537	9.453 279	0.006 154	0.012 103
0.293	0.018 381	9.473 064	9.454 683	0.006 196	0.012 185
0.294	0.018 505	9.474 586	9.456 081	0.006 239	0.012 267
0.295	0.018 629	9.476 103	9.457 474	0.006 281	0.012 348
0.296	0.018 754	9.477 615	9.458 861	0.006 323	0.012 431
0.297	0.018 879	9.479 123	9.460 243	0.006 366	0.012 513
0.298	0.019 005	9.480 625	9.461 620	0.006 409	0.012 596
0.299	0.019 131	9.482 123	9.462 992	0.006 452	0.012 679

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(*Continued*)

x	log cosh x	log sinh x	log tanh x	$\log \frac{\sinh x}{x}$	$\log \frac{x}{\tanh x}$
0.300	0.019 257	9.483 616	9.464 359	0.006 495	0.012 762
0.301	0.019 384	9.485 105	9.465 721	0.006 538	0.012 845
0.302	0.019 511	9.486 589	9.467 078	0.006 582	0.012 929
0.303	0.019 638	9.488 068	9.468 429	0.006 625	0.013 013
0.304	0.019 766	9.489 542	9.469 776	0.006 669	0.013 097
0.305	0.019 894	9.491 012	9.471 118	0.006 713	0.013 182
0.306	0.020 023	9.492 478	9.472 455	0.006 757	0.013 267
0.307	0.020 152	9.493 939	9.473 787	0.006 801	0.013 352
0.308	0.020 282	9.495 396	9.475 114	0.006 845	0.013 437
0.309	0.020 412	9.496 848	9.476 436	0.006 889	0.013 522
0.310	0.020 542	9.498 295	9.477 754	0.006 934	0.013 608
0.311	0.020 673	9.499 739	9.479 066	0.006 978	0.013 694
0.312	0.020 804	9.501 178	9.480 374	0.007 023	0.013 780
0.313	0.020 935	9.502 613	9.481 677	0.007 068	0.013 867
0.314	0.021 067	9.504 043	9.482 976	0.007 113	0.013 954
0.315	0.021 199	9.505 469	9.484 270	0.007 159	0.014 041
0.316	0.021 332	9.506 891	9.485 559	0.007 204	0.014 128
0.317	0.021 465	9.508 309	9.486 844	0.007 249	0.014 216
0.318	0.021 598	9.509 722	9.488 124	0.007 295	0.014 303
0.319	0.021 732	9.511 132	9.489 399	0.007 341	0.014 391
0.320	0.021 866	9.512 537	9.490 670	0.007 387	0.014 480
0.321	0.022 001	9.513 938	9.491 937	0.007 433	0.014 568
0.322	0.022 136	9.515 335	9.493 199	0.007 479	0.014 657
0.323	0.022 271	9.516 728	9.494 457	0.007 525	0.014 746
0.324	0.022 407	9.518 117	9.495 710	0.007 572	0.014 835
0.325	0.022 543	9.519 502	9.496 959	0.007 619	0.014 925
0.326	0.022 680	9.520 883	9.498 203	0.007 665	0.015 015
0.327	0.022 817	9.522 260	9.499 443	0.007 712	0.015 105
0.328	0.022 954	9.523 633	9.500 679	0.007 759	0.015 195
0.329	0.023 092	9.525 003	9.501 911	0.007 807	0.015 285
0.330	0.023 230	9.526 368	9.503 138	0.007 854	0.015 376
0.331	0.023 369	9.527 730	9.504 361	0.007 902	0.015 467
0.332	0.023 508	9.529 087	9.505 580	0.007 949	0.015 558
0.333	0.023 647	9.530 441	9.506 794	0.007 997	0.015 650
0.334	0.023 787	9.531 791	9.508 005	0.008 045	0.015 742
0.335	0.023 927	9.533 138	9.509 211	0.008 093	0.015 834
0.336	0.024 067	9.534 480	9.510 413	0.008 141	0.015 926
0.337	0.024 208	9.535 819	9.511 611	0.008 190	0.016 019
0.338	0.024 349	9.537 155	9.512 805	0.008 238	0.016 111
0.339	0.024 491	9.538 486	9.513 995	0.008 287	0.016 204
0.340	0.024 633	9.539 814	9.515 181	0.008 335	0.016 298
0.341	0.024 775	9.541 139	9.516 363	0.008 384	0.016 391
0.342	0.024 918	9.542 459	9.517 541	0.008 433	0.016 485
0.343	0.025 061	9.543 777	9.518 715	0.008 483	0.016 579
0.344	0.025 205	9.545 090	9.519 885	0.008 532	0.016 673
0.345	0.025 349	9.546 400	9.521 052	0.008 581	0.016 768
0.346	0.025 493	9.547 707	9.522 214	0.008 631	0.016 862
0.347	0.025 638	9.549 010	9.523 372	0.008 681	0.016 957
0.348	0.025 783	9.550 310	9.524 527	0.008 731	0.017 052
0.349	0.025 929	9.551 606	9.525 678	0.008 781	0.017 148

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS

(Continued)

x	log cosh x	log sinh x	log tanh x	$\log \frac{\sinh x}{x}$	$\log \frac{x}{\tanh x}$
0.350	0.026 075	9.552 899	9.526 824	0.008 831	0.017 244
0.351	0.026 221	9.554 188	9.527 968	0.008 881	0.017 340
0.352	0.026 367	9.555 474	9.529 107	0.008 932	0.017 436
0.353	0.026 515	9.556 757	9.530 242	0.008 982	0.017 532
0.354	0.026 662	9.558 036	9.531 374	0.009 033	0.017 629
0.355	0.026 810	9.559 312	9.532 503	0.009 084	0.017 726
0.356	0.026 958	9.560 585	9.533 627	0.009 135	0.017 823
0.357	0.027 107	9.561 854	9.534 748	0.009 186	0.017 920
0.358	0.027 256	9.563 121	9.535 865	0.009 238	0.018 018
0.359	0.027 405	9.564 383	9.536 979	0.009 289	0.018 116
0.360	0.027 555	9.565 643	9.538 088	0.009 341	0.018 214
0.361	0.027 705	9.566 900	9.539 195	0.009 392	0.018 312
0.362	0.027 855	9.568 153	9.540 298	0.009 444	0.018 411
0.363	0.028 006	9.569 403	9.541 397	0.009 496	0.018 510
0.364	0.028 157	9.570 650	9.542 492	0.009 548	0.018 609
0.365	0.028 309	9.571 894	9.543 585	0.009 601	0.018 708
0.366	0.028 461	9.573 134	9.544 673	0.009 653	0.018 808
0.367	0.028 613	9.574 372	9.545 758	0.009 706	0.018 908
0.368	0.028 766	9.575 606	9.546 840	0.009 758	0.019 008
0.369	0.028 919	9.576 838	9.547 918	0.009 811	0.019 108
0.370	0.029 073	9.578 066	9.548 993	0.009 864	0.019 209
0.371	0.029 227	9.579 291	9.550 065	0.009 917	0.019 309
0.372	0.029 381	9.580 514	9.551 133	0.009 971	0.019 410
0.373	0.029 536	9.581 733	9.552 197	0.010 024	0.019 512
0.374	0.029 691	9.582 949	9.553 258	0.010 078	0.019 613
0.375	0.029 846	9.584 163	9.554 316	0.010 131	0.019 715
0.376	0.030 002	9.585 373	9.555 371	0.010 185	0.019 817
0.377	0.030 158	9.586 581	9.556 422	0.010 239	0.019 919
0.378	0.030 315	9.587 785	9.557 470	0.010 293	0.020 022
0.379	0.030 472	9.588 987	9.558 515	0.010 348	0.020 124
0.380	0.030 629	9.590 186	9.559 556	0.010 402	0.020 227
0.381	0.030 787	9.591 382	9.560 595	0.010 457	0.020 330
0.382	0.030 945	9.592 575	9.561 630	0.010 511	0.020 434
0.383	0.031 104	9.593 765	9.562 662	0.010 566	0.020 537
0.384	0.031 262	9.594 952	9.563 690	0.010 621	0.020 641
0.385	0.031 422	9.596 137	9.564 716	0.010 676	0.020 745
0.386	0.031 581	9.597 319	9.565 738	0.010 732	0.020 850
0.387	0.031 741	9.598 498	9.566 757	0.010 787	0.020 954
0.388	0.031 901	9.599 674	9.567 773	0.010 843	0.021 059
0.389	0.032 062	9.600 848	9.568 786	0.010 898	0.021 164
0.390	0.032 223	9.602 019	9.569 795	0.010 954	0.021 269
0.391	0.032 385	9.603 187	9.570 802	0.011 010	0.021 375
0.392	0.032 547	9.604 352	9.571 806	0.011 066	0.021 480
0.393	0.032 709	9.605 515	9.572 806	0.011 122	0.021 586
0.394	0.032 871	9.606 675	9.573 804	0.011 179	0.021 693
0.395	0.033 034	9.607 832	9.574 798	0.011 235	0.021 799
0.396	0.033 198	9.608 987	9.575 789	0.011 292	0.021 906
0.397	0.033 361	9.610 139	9.576 778	0.011 349	0.022 013
0.398	0.033 525	9.611 289	9.577 763	0.011 406	0.022 120
0.399	0.033 690	9.612 436	9.578 746	0.011 463	0.022 227

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS

(Continued)

x	log cosh x	log sinh x	log tanh x	$\log \frac{\sinh x}{x}$	$\log \frac{x}{\tanh x}$
0.400	0.033 855	9.613 580	9.579 725	0.011 520	0.022 335
0.401	0.034 020	9.614 722	9.580 702	0.011 577	0.022 443
0.402	0.034 186	9.615 861	9.581 675	0.011 635	0.022 551
0.403	0.034 351	9.616 998	9.582 646	0.011 693	0.022 659
0.404	0.034 518	9.618 132	9.583 614	0.011 750	0.022 767
0.405	0.034 684	9.619 263	9.584 579	0.011 808	0.022 876
0.406	0.034 852	9.620 392	9.585 541	0.011 866	0.022 985
0.407	0.035 019	9.621 519	9.586 500	0.011 925	0.023 094
0.408	0.035 187	9.622 643	9.587 456	0.011 983	0.023 204
0.409	0.035 355	9.623 765	9.588 410	0.012 041	0.023 313
0.410	0.035 523	9.624 884	9.589 360	0.012 100	0.023 423
0.411	0.035 692	9.626 001	9.590 308	0.012 159	0.023 534
0.412	0.035 862	9.627 115	9.591 253	0.012 218	0.023 644
0.413	0.036 031	9.628 227	9.592 196	0.012 277	0.023 754
0.414	0.036 201	9.629 336	9.593 135	0.012 336	0.023 865
0.415	0.036 372	9.630 443	9.594 072	0.012 395	0.023 976
0.416	0.036 542	9.631 548	9.595 006	0.012 455	0.024 088
0.417	0.036 713	9.632 650	9.595 937	0.012 514	0.024 199
0.418	0.036 885	9.633 750	9.596 866	0.012 574	0.024 311
0.419	0.037 057	9.634 848	9.597 791	0.012 634	0.024 423
0.420	0.037 229	9.635 943	9.598 714	0.012 694	0.024 535
0.421	0.037 401	9.637 036	9.599 635	0.012 754	0.024 647
0.422	0.037 574	9.638 127	9.600 553	0.012 814	0.024 760
0.423	0.037 748	9.639 215	9.601 468	0.012 875	0.024 873
0.424	0.037 921	9.640 301	9.602 380	0.012 936	0.024 986
0.425	0.038 095	9.641 385	9.603 290	0.012 996	0.025 099
0.426	0.038 270	9.642 467	9.604 197	0.013 057	0.025 213
0.427	0.038 445	9.643 546	9.605 101	0.013 118	0.025 326
0.428	0.038 620	9.644 623	9.606 003	0.013 179	0.025 440
0.429	0.038 795	9.645 698	9.606 903	0.013 241	0.025 555
0.430	0.038 971	9.646 770	9.607 799	0.013 302	0.025 669
0.431	0.039 147	9.647 841	9.608 694	0.013 364	0.025 784
0.432	0.039 324	9.648 909	9.609 585	0.013 425	0.025 899
0.433	0.039 501	9.649 975	9.610 474	0.013 487	0.026 014
0.434	0.039 678	9.651 039	9.611 361	0.013 549	0.026 129
0.435	0.039 856	9.652 100	9.612 245	0.013 611	0.026 244
0.436	0.040 034	9.653 160	9.613 126	0.013 673	0.026 360
0.437	0.040 212	9.654 217	9.614 005	0.013 736	0.026 476
0.438	0.040 391	9.655 273	9.614 882	0.013 798	0.026 592
0.439	0.040 570	9.656 326	9.615 756	0.013 861	0.026 709
0.440	0.040 749	9.657 377	9.616 627	0.013 924	0.026 825
0.441	0.040 929	9.658 425	9.617 496	0.013 987	0.026 942
0.442	0.041 109	9.659 472	9.618 363	0.014 050	0.027 059
0.443	0.041 290	9.660 517	9.619 227	0.014 113	0.027 177
0.444	0.041 471	9.661 560	9.620 089	0.014 177	0.027 294
0.445	0.041 652	9.662 600	9.620 948	0.014 240	0.027 412
0.446	0.041 834	9.663 639	9.621 805	0.014 304	0.027 530
0.447	0.042 016	9.664 675	9.622 659	0.014 368	0.027 648
0.448	0.042 198	9.665 709	9.623 511	0.014 431	0.027 767
0.449	0.042 381	9.666 742	9.624 361	0.014 496	0.027 885

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

z	log cosh z	log sinh z	log tanh z	$\log \frac{\sinh z}{z}$	$\log \frac{z}{\tanh z}$
0.450	0.042 564	9.667 772	9.625 209	0.014 560	0.028 004
0.451	0.042 747	9.668 801	9.626 053	0.014 624	0.028 123
0.452	0.042 931	9.669 827	9.626 896	0.014 689	0.028 242
0.453	0.043 115	9.670 851	9.627 736	0.014 753	0.028 362
0.454	0.043 300	9.671 874	9.628 574	0.014 818	0.028 482
0.455	0.043 484	9.672 894	9.629 410	0.014 883	0.028 601
0.456	0.043 670	9.673 913	9.630 243	0.014 948	0.028 722
0.457	0.043 855	9.674 929	9.631 074	0.015 013	0.028 842
0.458	0.044 041	9.675 944	9.631 903	0.015 078	0.028 963
0.459	0.044 227	9.676 957	9.632 729	0.015 144	0.029 083
0.460	0.044 414	9.677 967	9.633 553	0.015 210	0.029 204
0.461	0.044 601	9.678 976	9.634 375	0.015 275	0.029 326
0.462	0.044 788	9.679 983	9.635 195	0.015 341	0.029 447
0.463	0.044 976	9.680 988	9.636 012	0.015 407	0.029 569
0.464	0.045 164	9.681 991	9.636 827	0.015 473	0.029 690
0.465	0.045 352	9.682 993	9.637 640	0.015 540	0.029 813
0.466	0.045 541	9.683 992	9.638 451	0.015 606	0.029 935
0.467	0.045 730	9.684 989	9.639 260	0.015 673	0.030 057
0.468	0.045 919	9.685 985	9.640 066	0.015 739	0.030 180
0.469	0.046 109	9.686 979	9.640 870	0.015 806	0.030 303
0.470	0.046 299	9.687 971	9.641 672	0.015 873	0.030 426
0.471	0.046 490	9.688 961	9.642 471	0.015 940	0.030 550
0.472	0.046 681	9.689 950	9.643 269	0.016 008	0.030 673
0.473	0.046 872	9.690 936	9.644 064	0.016 075	0.030 797
0.474	0.047 063	9.691 921	9.644 857	0.016 143	0.030 921
0.475	0.047 255	9.692 904	9.645 649	0.016 210	0.031 045
0.476	0.047 447	9.693 885	9.646 437	0.016 278	0.031 169
0.477	0.047 640	9.694 864	9.647 224	0.016 346	0.031 294
0.478	0.047 833	9.695 842	9.648 009	0.016 414	0.031 419
0.479	0.048 026	9.696 818	9.648 792	0.016 482	0.031 544
0.480	0.048 220	9.697 792	9.649 572	0.016 551	0.031 669
0.481	0.048 414	9.698 764	9.650 350	0.016 619	0.031 795
0.482	0.048 608	9.699 735	9.651 127	0.016 688	0.031 920
0.483	0.048 803	9.700 704	9.651 901	0.016 757	0.032 046
0.484	0.048 998	9.701 671	9.652 673	0.016 826	0.032 172
0.485	0.049 193	9.702 636	9.653 443	0.016 895	0.032 299
0.486	0.049 389	9.703 600	9.654 211	0.016 964	0.032 425
0.487	0.049 585	9.704 562	9.654 977	0.017 033	0.032 552
0.488	0.049 781	9.705 522	9.655 741	0.017 103	0.032 679
0.489	0.049 978	9.706 481	9.656 503	0.017 172	0.032 806
0.490	0.050 175	9.707 438	9.657 263	0.017 242	0.032 933
0.491	0.050 373	9.708 393	9.658 021	0.017 312	0.033 061
0.492	0.050 570	9.709 347	9.658 777	0.017 382	0.033 189
0.493	0.050 769	9.710 299	9.659 530	0.017 452	0.033 316
0.494	0.050 967	9.711 249	9.660 282	0.017 522	0.033 445
0.495	0.051 166	9.712 198	9.661 032	0.017 593	0.033 573
0.496	0.051 365	9.713 145	9.661 780	0.017 663	0.033 702
0.497	0.051 565	9.714 091	9.662 526	0.017 734	0.033 830
0.498	0.051 764	9.715 034	9.663 270	0.017 805	0.033 959
0.499	0.051 965	9.715 977	9.664 012	0.017 876	0.034 089

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
0.500	0.052 165	201	9.716 917	939	9.664 752	738
0.501	0.052 366	201	9.717 856	938	9.665 490	736
0.502	0.052 567	202	9.718 794	935	9.666 226	735
0.503	0.052 769	202	9.719 729	935	9.666 961	732
0.504	0.052 971	202	9.720 664	932	9.667 693	730
0.505	0.053 173	202	9.721 596	931	9.668 423	729
0.506	0.053 375	203	9.722 527	930	9.669 152	727
0.507	0.053 578	204	9.723 457	928	9.669 879	724
0.508	0.053 782	203	9.724 385	926	9.670 603	723
0.509	0.053 985	204	9.725 311	925	9.671 326	721
0.510	0.054 189	204	9.726 236	924	9.672 047	719
0.511	0.054 393	205	9.727 160	921	9.672 766	718
0.512	0.054 598	205	9.728 081	921	9.673 484	715
0.513	0.054 803	205	9.729 002	919	9.674 199	714
0.514	0.055 008	206	9.729 921	917	9.674 913	711
0.515	0.055 214	206	9.730 838	916	9.675 624	710
0.516	0.055 420	206	9.731 754	914	9.676 334	708
0.517	0.055 626	207	9.732 668	913	9.677 042	706
0.518	0.055 833	207	9.733 581	911	9.677 748	705
0.519	0.056 040	207	9.734 492	910	9.678 453	702
0.520	0.056 247	207	9.735 402	909	9.679 155	701
0.521	0.056 454	208	9.736 311	906	9.679 856	699
0.522	0.056 662	209	9.737 217	906	9.680 555	697
0.523	0.056 871	208	9.738 123	904	9.681 252	696
0.524	0.057 079	209	9.739 027	903	9.681 948	693
0.525	0.057 288	210	9.739 930	901	9.682 641	692
0.526	0.057 498	209	9.740 831	899	9.683 333	690
0.527	0.057 707	210	9.741 730	899	9.684 023	688
0.528	0.057 917	211	9.742 629	897	9.684 711	687
0.529	0.058 128	210	9.743 526	895	9.685 398	685
0.530	0.058 338	211	9.744 421	894	9.686 083	683
0.531	0.058 549	211	9.745 315	893	9.686 766	681
0.532	0.058 760	212	9.746 208	891	9.687 447	680
0.533	0.058 972	212	9.747 099	890	9.688 127	678
0.534	0.059 184	212	9.747 989	888	9.688 805	676
0.535	0.059 396	213	9.748 877	887	9.689 481	674
0.536	0.059 609	213	9.749 764	886	9.690 155	673
0.537	0.059 822	213	9.750 650	884	9.690 828	671
0.538	0.060 035	214	9.751 534	883	9.691 499	670
0.539	0.060 249	214	9.752 417	882	9.692 169	667
0.540	0.060 463	214	9.753 299	880	9.692 836	666
0.541	0.060 677	215	9.754 179	879	9.693 502	665
0.542	0.060 892	214	9.755 058	878	9.694 167	662
0.543	0.061 106	216	9.755 936	876	9.694 829	661
0.544	0.061 322	215	9.756 812	875	9.695 490	660
0.545	0.061 537	216	9.757 687	874	9.696 150	657
0.546	0.061 753	216	9.758 561	872	9.696 807	656
0.547	0.061 969	217	9.759 433	871	9.697 463	655
0.548	0.062 186	217	9.760 304	869	9.698 118	652
0.549	0.062 403	217	9.761 173	869	9.698 770	652

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(*Continued*)

x	log cosh x	d	log sinh x	d	log tanh x	d
0.550	0.062 620	218	9.762 042	867	9.699 422	649
0.551	0.062 838	218	9.762 909	866	9.700 071	648
0.552	0.063 056	218	9.763 775	864	9.700 719	646
0.553	0.063 274	218	9.764 639	863	9.701 365	645
0.554	0.063 492	219	9.765 502	862	9.702 010	643
0.555	0.063 711	219	9.766 364	861	9.702 653	641
0.556	0.063 930	220	9.767 225	859	9.703 294	640
0.557	0.064 150	219	9.768 084	858	9.703 934	639
0.558	0.064 369	221	9.768 942	857	9.704 573	636
0.559	0.064 590	220	9.769 799	855	9.705 209	635
0.560	0.064 810	221	9.770 654	855	9.705 844	634
0.561	0.065 031	221	9.771 509	853	9.706 478	632
0.562	0.065 252	221	9.772 362	852	9.707 110	630
0.563	0.065 473	222	9.773 214	850	9.707 740	629
0.564	0.065 695	222	9.774 064	850	9.708 369	628
0.565	0.065 917	223	9.774 914	848	9.708 997	625
0.566	0.066 140	222	9.775 762	847	9.709 622	625
0.567	0.066 362	223	9.776 609	846	9.710 247	622
0.568	0.066 585	224	9.777 455	844	9.710 869	621
0.569	0.066 809	223	9.778 299	843	9.711 490	620
0.570	0.067 032	224	9.779 142	842	9.712 110	618
0.571	0.067 256	225	9.779 984	841	9.712 728	617
0.572	0.067 481	224	9.780 825	840	9.713 345	615
0.573	0.067 705	225	9.781 665	839	9.713 960	614
0.574	0.067 930	225	9.782 504	837	9.714 574	612
0.575	0.068 155	226	9.783 341	836	9.715 186	610
0.576	0.068 381	226	9.784 177	835	9.715 796	609
0.577	0.068 607	226	9.785 012	834	9.716 405	608
0.578	0.068 833	226	9.785 846	833	9.717 013	606
0.579	0.069 059	227	9.786 679	831	9.717 619	605
0.580	0.069 286	227	9.787 510	830	9.718 224	603
0.581	0.069 513	228	9.788 340	830	9.718 827	602
0.582	0.069 741	228	9.789 170	828	9.719 429	600
0.583	0.069 969	228	9.789 998	827	9.720 029	599
0.584	0.070 197	228	9.790 825	825	9.720 628	597
0.585	0.070 425	229	9.791 650	825	9.721 225	596
0.586	0.070 654	229	9.792 475	823	9.721 821	595
0.587	0.070 883	229	9.793 298	823	9.722 416	593
0.588	0.071 112	230	9.794 121	821	9.723 009	591
0.589	0.071 342	230	9.794 942	820	9.723 600	590
0.590	0.071 572	230	9.795 762	819	9.724 190	589
0.591	0.071 802	231	9.796 581	818	9.724 779	587
0.592	0.072 033	231	9.797 399	817	9.725 366	586
0.593	0.072 264	231	9.798 216	816	9.725 952	585
0.594	0.072 495	231	9.799 032	814	9.726 537	583
0.595	0.072 726	232	9.799 846	814	9.727 120	582
0.596	0.072 958	232	9.800 660	812	9.727 702	580
0.597	0.073 190	233	9.801 472	812	9.728 282	579
0.598	0.073 423	233	9.802 284	810	9.728 861	577
0.599	0.073 656	233	9.803 094	809	9.729 438	576

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
0.600	0.073 889	233	9.803 903	808	9.730 014	575
0.601	0.074 122	234	9.804 711	807	9.730 589	573
0.602	0.074 356	234	9.805 518	806	9.731 162	572
0.603	0.074 590	234	9.806 324	805	9.731 734	571
0.604	0.074 824	235	9.807 129	804	9.732 305	569
0.605	0.075 059	235	9.807 933	803	9.732 874	568
0.606	0.075 294	235	9.808 736	802	9.733 442	567
0.607	0.075 529	236	9.809 538	801	9.734 009	565
0.608	0.075 765	235	9.810 339	799	9.734 574	564
0.609	0.076 000	237	9.811 138	799	9.735 138	562
0.610	0.076 237	236	9.811 937	798	9.735 700	562
0.611	0.076 473	237	9.812 735	796	9.736 262	559
0.612	0.076 710	237	9.813 531	796	9.736 821	559
0.613	0.076 947	237	9.814 327	794	9.737 380	557
0.614	0.077 184	238	9.815 121	794	9.737 937	556
0.615	0.077 422	238	9.815 915	792	9.738 493	554
0.616	0.077 660	238	9.816 707	792	9.739 047	554
0.617	0.077 898	239	9.817 499	790	9.739 601	552
0.618	0.078 137	239	9.818 289	790	9.740 153	550
0.619	0.078 376	239	9.819 079	788	9.740 703	549
0.620	0.078 615	239	9.819 867	788	9.741 252	549
0.621	0.078 854	240	9.820 655	786	9.741 801	546
0.622	0.079 094	240	9.821 441	786	9.742 347	546
0.623	0.079 334	241	9.822 227	785	9.742 893	544
0.624	0.079 575	240	9.823 012	783	9.743 437	543
0.625	0.079 815	241	9.823 795	783	9.743 980	541
0.626	0.080 056	242	9.824 578	781	9.744 521	541
0.627	0.080 298	241	9.825 359	781	9.745 062	539
0.628	0.080 539	242	9.826 140	780	9.745 601	537
0.629	0.080 781	243	9.826 920	778	9.746 138	537
0.630	0.081 024	242	9.827 698	778	9.746 675	535
0.631	0.081 266	243	9.828 476	777	9.747 210	534
0.632	0.081 509	243	9.829 253	776	9.747 744	533
0.633	0.081 752	243	9.830 029	775	9.748 277	531
0.634	0.081 995	244	9.830 804	774	9.748 808	531
0.635	0.082 239	244	9.831 578	773	9.749 339	529
0.636	0.082 483	244	9.832 351	772	9.749 868	527
0.637	0.082 727	245	9.833 123	771	9.750 395	527
0.638	0.082 972	245	9.833 894	770	9.750 922	525
0.639	0.083 217	245	9.834 664	769	9.751 447	524
0.640	0.083 462	245	9.835 433	769	9.751 971	523
0.641	0.083 707	246	9.836 202	767	9.752 494	522
0.642	0.083 953	246	9.836 969	766	9.753 016	520
0.643	0.084 199	247	9.837 735	766	9.753 536	519
0.644	0.084 446	246	9.838 501	765	9.754 055	518
0.645	0.084 692	247	9.839 266	763	9.754 573	517
0.646	0.084 939	248	9.840 029	763	9.755 090	516
0.647	0.085 187	247	9.840 792	762	9.755 606	514
0.648	0.085 434	248	9.841 554	761	9.756 120	513
0.649	0.085 682	248	9.842 315	760	9.756 633	512

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
0.650	0.085 930	248	9.843 075	760	9.757 145	511
0.651	0.086 178	249	9.843 835	758	9.757 656	510
0.652	0.086 427	249	9.844 593	758	9.758 166	508
0.653	0.086 676	249	9.845 351	756	9.758 674	508
0.654	0.086 925	250	9.846 107	756	9.759 182	506
0.655	0.087 175	250	9.846 863	755	9.759 688	505
0.656	0.087 425	250	9.847 618	754	9.760 193	503
0.657	0.087 675	251	9.848 372	753	9.760 696	503
0.658	0.087 926	250	9.849 125	752	9.761 199	502
0.659	0.088 176	251	9.849 877	751	9.761 701	500
0.660	0.088 427	252	9.850 628	751	9.762 201	499
0.661	0.088 679	251	9.851 379	749	9.762 700	498
0.662	0.088 930	252	9.852 128	749	9.763 198	497
0.663	0.089 182	252	9.852 877	748	9.763 695	496
0.664	0.089 434	253	9.853 625	747	9.764 191	494
0.665	0.089 687	253	9.854 372	746	9.764 685	494
0.666	0.089 940	253	9.855 118	745	9.765 179	492
0.667	0.090 193	253	9.855 863	745	9.765 671	491
0.668	0.090 446	254	9.856 608	744	9.766 162	490
0.669	0.090 700	254	9.857 352	742	9.766 652	489
0.670	0.090 954	254	9.858 094	742	9.767 141	488
0.671	0.091 208	254	9.858 836	742	9.767 629	486
0.672	0.091 462	255	9.859 578	740	9.768 115	486
0.673	0.091 717	255	9.860 318	739	9.768 601	484
0.674	0.091 972	255	9.861 057	739	9.769 085	484
0.675	0.092 227	256	9.861 796	738	9.769 569	482
0.676	0.092 483	256	9.862 534	737	9.770 051	481
0.677	0.092 739	256	9.863 271	736	9.770 532	480
0.678	0.092 995	257	9.864 007	736	9.771 012	479
0.679	0.093 252	256	9.864 743	734	9.771 491	478
0.680	0.093 508	257	9.865 477	734	9.771 969	477
0.681	0.093 765	258	9.866 211	733	9.772 446	475
0.682	0.094 023	257	9.866 944	732	9.772 921	475
0.683	0.094 280	258	9.867 676	732	9.773 396	474
0.684	0.094 538	258	9.868 408	730	9.773 870	472
0.685	0.094 796	259	9.869 138	730	9.774 342	471
0.686	0.095 055	258	9.869 868	729	9.774 813	471
0.687	0.095 313	259	9.870 597	728	9.775 284	469
0.688	0.095 572	260	9.871 325	728	9.775 753	468
0.689	0.095 832	259	9.872 053	726	9.776 221	467
0.690	0.096 091	260	9.872 779	726	9.776 688	466
0.691	0.096 351	260	9.873 505	725	9.777 154	465
0.692	0.096 611	261	9.874 230	725	9.777 619	464
0.693	0.096 872	260	9.874 955	723	9.778 083	463
0.694	0.097 132	261	9.875 678	723	9.778 546	462
0.695	0.097 393	262	9.876 401	722	9.779 008	461
0.696	0.097 655	261	9.877 123	721	9.779 469	459
0.697	0.097 916	262	9.877 844	721	9.779 928	459
0.698	0.098 178	262	9.878 565	720	9.780 387	458
0.699	0.098 440	262	9.879 285	719	9.780 845	456

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
0.700	0.098 702	263	9.880 004	718	9.781 301	456
0.701	0.098 965	263	9.880 722	717	9.781 757	454
0.702	0.099 228	263	9.881 439	717	9.782 211	454
0.703	0.099 491	263	9.882 156	716	9.782 665	453
0.704	0.099 754	264	9.882 872	715	9.783 118	451
0.705	0.100 018	264	9.883 587	715	9.783 569	451
0.706	0.100 282	264	9.884 302	713	9.784 020	449
0.707	0.100 546	265	9.885 015	713	9.784 469	449
0.708	0.100 811	265	9.885 728	713	9.784 918	447
0.709	0.101 076	265	9.886 441	711	9.785 365	446
0.710	0.101 341	265	9.887 152	711	9.785 811	446
0.711	0.101 606	266	9.887 863	710	9.786 257	444
0.712	0.101 872	266	9.888 573	709	9.786 701	444
0.713	0.102 138	266	9.889 282	709	9.787 145	442
0.714	0.102 404	266	9.889 991	708	9.787 587	442
0.715	0.102 670	267	9.890 699	707	9.788 029	440
0.716	0.102 937	267	9.891 406	707	9.788 469	440
0.717	0.103 204	267	9.892 113	705	9.788 909	438
0.718	0.103 471	268	9.892 818	705	9.789 347	438
0.719	0.103 739	267	9.893 523	705	9.789 785	436
0.720	0.104 006	268	9.894 228	703	9.790 221	436
0.721	0.104 274	269	9.894 931	703	9.790 657	434
0.722	0.104 543	268	9.895 634	702	9.791 091	434
0.723	0.104 811	269	9.896 336	702	9.791 525	433
0.724	0.105 080	269	9.897 038	701	9.791 958	431
0.725	0.105 349	270	9.897 739	700	9.792 389	431
0.726	0.105 619	269	9.898 439	699	9.792 820	430
0.727	0.105 888	270	9.899 138	699	9.793 250	429
0.728	0.106 158	271	9.899 837	698	9.793 679	428
0.729	0.106 429	270	9.900 535	698	9.794 107	427
0.730	0.106 699	271	9.901 233	696	9.794 534	426
0.731	0.106 970	271	9.901 929	696	9.794 960	425
0.732	0.107 241	271	9.902 625	696	9.795 385	424
0.733	0.107 512	271	9.903 321	694	9.795 809	423
0.734	0.107 783	272	9.904 015	694	9.796 232	422
0.735	0.108 055	272	9.904 709	693	9.796 654	421
0.736	0.108 327	273	9.905 402	693	9.797 075	421
0.737	0.108 600	272	9.906 095	692	9.797 496	419
0.738	0.108 872	273	9.906 787	691	9.797 915	418
0.739	0.109 145	273	9.907 478	691	9.798 333	418
0.740	0.109 418	273	9.908 169	690	9.798 751	416
0.741	0.109 691	274	9.908 859	689	9.799 167	416
0.742	0.109 965	274	9.909 548	689	9.799 583	415
0.743	0.110 239	274	9.910 237	688	9.799 998	414
0.744	0.110 513	275	9.910 925	687	9.800 412	413
0.745	0.110 788	274	9.911 612	687	9.800 825	412
0.746	0.111 062	275	9.912 299	686	9.801 237	411
0.747	0.111 337	275	9.912 985	685	9.801 648	410
0.748	0.111 612	276	9.913 670	685	9.802 058	409
0.749	0.111 888	276	9.914 355	684	9.802 467	409

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
0.750	0.112 164	276	9.915 039	684	9.802 876	407
0.751	0.112 440	276	9.915 723	682	9.803 283	407
0.752	0.112 716	276	9.916 405	683	9.803 690	405
0.753	0.112 992	277	9.917 088	681	9.804 095	405
0.754	0.113 269	277	9.917 769	681	9.804 500	404
0.755	0.113 546	277	9.918 450	680	9.804 904	403
0.756	0.113 823	278	9.919 130	680	9.805 307	402
0.757	0.114 101	278	9.919 810	679	9.805 709	401
0.758	0.114 379	278	9.920 489	678	9.806 110	401
0.759	0.114 657	278	9.921 167	678	9.806 511	399
0.760	0.114 935	278	9.921 845	677	9.806 910	399
0.761	0.115 213	279	9.922 522	677	9.807 309	397
0.762	0.115 492	279	9.923 199	676	9.807 706	397
0.763	0.115 771	280	9.923 875	675	9.808 103	396
0.764	0.116 051	279	9.924 550	675	9.808 499	395
0.765	0.116 330	280	9.925 225	674	9.808 894	395
0.766	0.116 610	280	9.925 899	673	9.809 289	393
0.767	0.116 890	280	9.926 572	673	9.809 682	392
0.768	0.117 170	281	9.927 245	672	9.810 074	392
0.769	0.117 451	281	9.927 917	672	9.810 466	391
0.770	0.117 732	281	9.928 589	671	9.810 857	390
0.771	0.118 013	281	9.929 260	670	9.811 247	389
0.772	0.118 294	282	9.929 930	670	9.811 636	388
0.773	0.118 576	282	9.930 600	669	9.812 024	388
0.774	0.118 858	282	9.931 269	669	9.812 412	386
0.775	0.119 140	282	9.931 938	668	9.812 798	386
0.776	0.119 422	283	9.932 606	667	9.813 184	385
0.777	0.119 705	282	9.933 273	667	9.813 569	384
0.778	0.119 987	284	9.933 940	666	9.813 953	383
0.779	0.120 271	283	9.934 606	666	9.814 336	382
0.780	0.120 554	283	9.935 272	665	9.814 718	382
0.781	0.120 837	284	9.935 937	665	9.815 100	380
0.782	0.121 121	284	9.936 602	663	9.815 480	380
0.783	0.121 405	285	9.937 265	664	9.815 860	379
0.784	0.121 690	284	9.937 929	663	9.816 239	378
0.785	0.121 974	285	9.938 592	662	9.816 617	378
0.786	0.122 259	285	9.939 254	661	9.816 995	376
0.787	0.122 544	286	9.939 915	661	9.817 371	376
0.788	0.122 830	285	9.940 576	661	9.817 747	375
0.789	0.123 115	286	9.941 237	660	9.818 122	374
0.790	0.123 401	286	9.941 897	659	9.818 496	373
0.791	0.123 687	286	9.942 556	659	9.818 869	372
0.792	0.123 973	287	9.943 215	658	9.819 241	372
0.793	0.124 260	287	9.943 873	658	9.819 613	371
0.794	0.124 547	287	9.944 531	657	9.819 984	370
0.795	0.124 834	287	9.945 188	656	9.820 354	369
0.796	0.125 121	288	9.945 844	656	9.820 723	369
0.797	0.125 409	287	9.946 500	656	9.821 092	367
0.798	0.125 696	288	9.947 156	654	9.821 459	367
0.799	0.125 984	289	9.947 810	655	9.821 826	366

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(*Continued*)

x	log cosh x	d	log sinh x	d	log tanh x	d
0.800	0.126 273	288	9.948 465	654	9.822 192	365
0.801	0.126 561	289	9.949 119	653	9.822 557	365
0.802	0.126 850	289	9.949 772	652	9.822 922	363
0.803	0.127 139	289	9.950 424	652	9.823 285	363
0.804	0.127 428	290	9.951 076	652	9.823 648	362
0.805	0.127 718	289	9.951 728	651	9.824 010	362
0.806	0.128 007	290	9.952 379	651	9.824 372	360
0.807	0.128 297	291	9.953 030	649	9.824 732	360
0.808	0.128 588	290	9.953 679	650	9.825 092	359
0.809	0.128 878	291	9.954 329	649	9.825 451	358
0.810	0.129 169	291	9.954 978	648	9.825 809	358
0.811	0.129 460	291	9.955 626	648	9.826 167	356
0.812	0.129 751	291	9.956 274	647	9.826 523	356
0.813	0.130 042	292	9.956 921	647	9.826 879	355
0.814	0.130 334	292	9.957 568	646	9.827 234	354
0.815	0.130 626	292	9.958 214	646	9.827 588	354
0.816	0.130 918	292	9.958 860	645	9.827 942	353
0.817	0.131 210	293	9.959 505	645	9.828 295	352
0.818	0.131 503	292	9.960 150	644	9.828 647	351
0.819	0.131 795	294	9.960 794	643	9.828 998	351
0.820	0.132 089	293	9.961 437	643	9.829 349	350
0.821	0.132 382	293	9.962 080	643	9.829 699	349
0.822	0.132 675	294	9.962 723	642	9.830 048	348
0.823	0.132 969	294	9.963 365	641	9.830 396	347
0.824	0.133 263	294	9.964 006	641	9.830 743	347
0.825	0.133 557	295	9.964 647	641	9.831 090	346
0.826	0.133 852	295	9.965 288	640	9.831 436	345
0.827	0.134 147	295	9.965 928	639	9.831 781	345
0.828	0.134 442	295	9.966 567	639	9.832 126	344
0.829	0.134 737	295	9.967 206	639	9.832 470	343
0.830	0.135 032	296	9.967 845	638	9.832 813	342
0.831	0.135 328	296	9.968 483	637	9.833 155	342
0.832	0.135 624	296	9.969 120	637	9.833 497	341
0.833	0.135 920	296	9.969 757	637	9.833 838	340
0.834	0.136 216	297	9.970 394	636	9.834 178	339
0.835	0.136 513	296	9.971 030	635	9.834 517	339
0.836	0.136 809	297	9.971 665	635	9.834 856	338
0.837	0.137 106	298	9.972 300	635	9.835 194	337
0.838	0.137 404	297	9.972 935	634	9.835 531	336
0.839	0.137 701	298	9.973 569	633	9.835 867	336
0.840	0.137 999	298	9.974 202	633	9.836 203	335
0.841	0.138 297	298	9.974 835	633	9.836 538	335
0.842	0.138 595	299	9.975 468	632	9.836 873	333
0.843	0.138 894	298	9.976 100	631	9.837 206	333
0.844	0.139 192	299	9.976 731	631	9.837 539	332
0.845	0.139 491	299	9.977 362	631	9.837 871	332
0.846	0.139 790	300	9.977 993	630	9.838 203	331
0.847	0.140 090	299	9.978 623	630	9.838 534	330
0.848	0.140 389	300	9.979 253	629	9.838 864	329
0.849	0.140 689	300	9.979 882	629	9.839 193	329

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(*Continued*)

x	log cosh x	d	log sinh x	d	log tanh x	d
0.850	0.140 989	300	9.980 511	628	9.839 522	328
0.851	0.141 289	301	9.981 139	627	9.839 850	327
0.852	0.141 590	300	9.981 766	628	9.840 177	326
0.853	0.141 890	301	9.982 394	627	9.840 503	326
0.854	0.142 191	301	9.983 021	626	9.840 829	326
0.855	0.142 492	302	9.983 647	626	9.841 155	324
0.856	0.142 794	301	9.984 273	625	9.841 479	324
0.857	0.143 095	302	9.984 898	625	9.841 803	323
0.858	0.143 397	302	9.985 523	624	9.842 126	322
0.859	0.143 699	302	9.986 147	624	9.842 448	322
0.860	0.144 001	303	9.986 771	624	9.842 770	321
0.861	0.144 304	303	9.987 395	623	9.843 091	320
0.862	0.144 607	303	9.988 018	623	9.843 411	320
0.863	0.144 910	303	9.988 641	622	9.843 731	319
0.864	0.145 213	303	9.989 263	621	9.844 050	318
0.865	0.145 516	304	9.989 884	622	9.844 368	318
0.866	0.145 820	304	9.990 506	620	9.844 686	317
0.867	0.146 124	304	9.991 126	621	9.845 003	316
0.868	0.146 428	304	9.991 747	620	9.845 319	316
0.869	0.146 732	304	9.992 367	619	9.845 635	315
0.870	0.147 036	305	9.992 986	619	9.845 950	314
0.871	0.147 341	305	9.993 605	619	9.846 264	314
0.872	0.147 646	305	9.994 224	618	9.846 578	313
0.873	0.147 951	306	9.994 842	617	9.846 891	312
0.874	0.148 257	305	9.995 459	618	9.847 203	311
0.875	0.148 562	306	9.996 077	616	9.847 514	311
0.876	0.148 868	306	9.996 693	617	9.847 825	311
0.877	0.149 174	306	9.997 310	616	9.848 136	309
0.878	0.149 480	307	9.997 926	615	9.848 445	309
0.879	0.149 787	306	9.998 541	615	9.848 754	309
0.880	0.150 093	307	9.999 156	615	9.849 063	307
0.881	0.150 400	307	9.999 771	614	9.849 370	307
0.882	0.150 707	308	0.000 385	613	9.849 677	307
0.883	0.151 015	307	0.000 998	614	9.849 984	305
0.884	0.151 322	308	0.001 612	612	9.850 289	305
0.885	0.151 630	308	0.002 224	613	9.850 594	305
0.886	0.151 938	308	0.002 837	612	9.850 899	304
0.887	0.152 246	309	0.003 449	611	9.851 203	303
0.888	0.152 555	308	0.004 060	611	9.851 506	302
0.889	0.152 863	309	0.004 671	611	9.851 808	302
0.890	0.153 172	309	0.005 282	610	9.852 110	301
0.891	0.153 481	309	0.005 892	610	9.852 411	301
0.892	0.153 790	310	0.006 502	610	9.852 712	300
0.893	0.154 100	310	0.007 112	609	9.853 012	299
0.894	0.154 410	310	0.007 721	608	9.853 311	299
0.895	0.154 720	310	0.008 329	609	9.853 610	298
0.896	0.155 030	310	0.008 938	607	9.853 908	297
0.897	0.155 340	311	0.009 545	608	9.854 205	297
0.898	0.155 651	310	0.010 153	607	9.854 502	296
0.899	0.155 961	311	0.010 760	606	9.854 798	296

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS

(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
0.900	0.156 272	312	0.011 366	606	9.855 094	295
0.901	0.156 584	311	0.011 972	606	9.855 389	294
0.902	0.156 895	312	0.012 578	605	9.855 683	294
0.903	0.157 207	311	0.013 183	605	9.855 977	293
0.904	0.157 518	312	0.013 788	604	9.856 270	292
0.905	0.157 830	313	0.014 392	604	9.856 562	292
0.906	0.158 143	312	0.014 996	604	9.856 854	291
0.907	0.158 455	313	0.015 600	603	9.857 145	291
0.908	0.158 768	313	0.016 203	603	9.857 436	290
0.909	0.159 081	313	0.016 806	603	9.857 726	289
0.910	0.159 394	313	0.017 409	602	9.858 015	289
0.911	0.159 707	314	0.018 011	601	9.858 304	288
0.912	0.160 021	313	0.018 612	602	9.858 592	287
0.913	0.160 334	314	0.019 214	600	9.858 879	287
0.914	0.160 648	314	0.019 814	601	9.859 166	287
0.915	0.160 962	315	0.020 415	600	9.859 453	285
0.916	0.161 277	314	0.021 015	600	9.859 738	285
0.917	0.161 591	315	0.021 615	599	9.860 023	285
0.918	0.161 906	315	0.022 214	599	9.860 308	284
0.919	0.162 221	315	0.022 813	598	9.860 592	283
0.920	0.162 536	315	0.023 411	598	9.860 875	283
0.921	0.162 851	316	0.024 009	598	9.861 158	282
0.922	0.163 167	316	0.024 607	597	9.861 440	282
0.923	0.163 483	316	0.025 204	597	9.861 722	281
0.924	0.163 799	316	0.025 801	597	9.862 003	280
0.925	0.164 115	316	0.026 398	596	9.862 283	280
0.926	0.164 431	317	0.026 994	596	9.862 563	279
0.927	0.164 748	317	0.027 590	595	9.862 842	279
0.928	0.165 065	317	0.028 185	595	9.863 121	278
0.929	0.165 382	317	0.028 780	595	9.863 399	277
0.930	0.165 699	317	0.029 375	594	9.863 676	277
0.931	0.166 016	318	0.029 969	594	9.863 953	276
0.932	0.166 334	317	0.030 563	593	9.864 229	276
0.933	0.166 651	318	0.031 156	593	9.864 505	275
0.934	0.166 969	319	0.031 749	593	9.864 780	275
0.935	0.167 288	318	0.032 342	593	9.865 055	273
0.936	0.167 606	319	0.032 935	592	9.865 328	274
0.937	0.167 925	318	0.033 527	591	9.865 602	273
0.938	0.168 243	320	0.034 118	591	9.865 875	272
0.939	0.168 563	319	0.034 709	591	9.866 147	272
0.940	0.168 882	319	0.035 300	591	9.866 419	271
0.941	0.169 201	320	0.035 891	590	9.866 690	270
0.942	0.169 521	320	0.036 481	590	9.866 960	270
0.943	0.169 841	320	0.037 071	589	9.867 230	270
0.944	0.170 161	320	0.037 660	589	9.867 500	269
0.945	0.170 481	320	0.038 249	589	9.867 769	268
0.946	0.170 801	321	0.038 838	588	9.868 037	268
0.947	0.171 122	320	0.039 426	588	9.868 305	267
0.948	0.171 442	321	0.040 014	588	9.868 572	266
0.949	0.171 763	322	0.040 602	587	9.868 838	266

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(*Continued*)

x	log cosh x	d	log sinh x	d	log tanh x	d
0.950	0.172 085	321	0.041 189	587	9.869 104	266
0.951	0.172 406	322	0.041 776	586	9.869 370	265
0.952	0.172 728	321	0.042 362	587	9.869 635	264
0.953	0.173 049	322	0.042 949	585	9.869 899	264
0.954	0.173 371	323	0.043 534	586	9.870 163	263
0.955	0.173 694	322	0.044 120	585	9.870 426	263
0.956	0.174 016	322	0.044 705	585	9.870 689	262
0.957	0.174 338	323	0.045 290	584	9.870 951	262
0.958	0.174 661	323	0.045 874	584	9.871 213	261
0.959	0.174 984	323	0.046 458	584	9.871 474	261
0.960	0.175 307	324	0.047 042	583	9.871 735	260
0.961	0.175 631	323	0.047 625	583	9.871 995	259
0.962	0.175 954	324	0.048 208	583	9.872 254	259
0.963	0.176 278	324	0.048 791	582	9.872 513	258
0.964	0.176 602	324	0.049 373	582	9.872 771	258
0.965	0.176 926	324	0.049 955	582	9.873 029	257
0.966	0.177 250	325	0.050 537	581	9.873 286	257
0.967	0.177 575	324	0.051 118	581	9.873 543	256
0.968	0.177 899	325	0.051 699	580	9.873 799	256
0.969	0.178 224	325	0.052 279	581	9.874 055	255
0.970	0.178 549	326	0.052 860	579	9.874 310	255
0.971	0.178 875	325	0.053 439	580	9.874 565	254
0.972	0.179 200	326	0.054 019	579	9.874 819	254
0.973	0.179 526	325	0.054 598	579	9.875 073	253
0.974	0.179 851	326	0.055 177	579	9.875 326	252
0.975	0.180 177	327	0.055 756	578	9.875 578	252
0.976	0.180 504	326	0.056 334	578	9.875 830	252
0.977	0.180 830	327	0.056 912	577	9.876 082	251
0.978	0.181 157	326	0.057 489	577	9.876 333	250
0.979	0.181 483	327	0.058 066	577	9.876 583	250
0.980	0.181 810	327	0.058 643	577	9.876 833	249
0.981	0.182 137	328	0.059 220	576	9.877 082	249
0.982	0.182 465	327	0.059 796	576	9.877 331	249
0.983	0.182 792	328	0.060 372	575	9.877 580	247
0.984	0.183 120	328	0.060 947	576	9.877 827	248
0.985	0.183 448	328	0.061 523	575	9.878 075	247
0.986	0.183 776	328	0.062 098	574	9.878 322	246
0.987	0.184 104	329	0.062 672	574	9.878 568	246
0.988	0.184 433	328	0.063 246	574	9.878 814	245
0.989	0.184 761	329	0.063 820	574	9.879 059	245
0.990	0.185 090	329	0.064 394	573	9.879 304	244
0.991	0.185 419	329	0.064 967	573	9.879 548	244
0.992	0.185 748	330	0.065 540	573	9.879 792	243
0.993	0.186 078	329	0.066 113	572	9.880 035	243
0.994	0.186 407	330	0.066 685	572	9.880 278	242
0.995	0.186 737	330	0.067 257	572	9.880 520	242
0.996	0.187 067	330	0.067 829	571	9.880 762	241
0.997	0.187 397	330	0.068 400	571	9.881 003	241
0.998	0.187 727	331	0.068 971	571	9.881 244	240
0.999	0.188 058	331	0.069 542	570	9.881 484	240

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(*Continued*)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.000	0.188 389	330	0.070 112	570	9.881 724	239
1.001	0.188 719	331	0.070 682	570	9.881 963	239
1.002	0.189 050	332	0.071 252	570	9.882 202	238
1.003	0.189 382	331	0.071 822	569	9.882 440	238
1.004	0.189 713	332	0.072 391	569	9.882 678	237
1.005	0.190 045	331	0.072 960	568	9.882 915	237
1.006	0.190 376	332	0.073 528	568	9.883 152	236
1.007	0.190 708	332	0.074 096	568	9.883 388	236
1.008	0.191 040	333	0.074 664	568	9.883 624	235
1.009	0.191 373	332	0.075 232	567	9.883 859	235
1.010	0.191 705	333	0.075 799	567	9.884 094	234
1.011	0.192 038	333	0.076 366	567	9.884 328	234
1.012	0.192 371	333	0.076 933	566	9.884 562	233
1.013	0.192 704	333	0.077 499	566	9.884 795	233
1.014	0.193 037	333	0.078 065	566	9.885 028	233
1.015	0.193 370	334	0.078 631	565	9.885 261	232
1.016	0.193 704	334	0.079 196	566	9.885 493	231
1.017	0.194 038	334	0.079 762	564	9.885 724	231
1.018	0.194 372	334	0.080 326	565	9.885 955	230
1.019	0.194 706	334	0.080 891	564	9.886 185	230
1.020	0.195 040	334	0.081 455	564	9.886 415	230
1.021	0.195 374	335	0.082 019	564	9.886 645	229
1.022	0.195 709	335	0.082 583	563	9.886 874	229
1.023	0.196 044	335	0.083 146	563	9.887 103	228
1.024	0.196 379	335	0.083 709	563	9.887 331	227
1.025	0.196 714	335	0.084 272	563	9.887 558	228
1.026	0.197 049	336	0.084 835	562	9.887 786	226
1.027	0.197 385	335	0.085 397	562	9.888 012	226
1.028	0.197 720	336	0.085 959	561	9.888 238	226
1.029	0.198 056	336	0.086 520	562	9.888 464	226
1.030	0.198 392	336	0.087 082	561	9.888 690	224
1.031	0.198 728	337	0.087 643	560	9.888 914	225
1.032	0.199 065	336	0.088 203	561	9.889 139	224
1.033	0.199 401	337	0.088 764	560	9.889 363	223
1.034	0.199 738	337	0.089 324	560	9.889 586	223
1.035	0.200 075	337	0.089 884	559	9.889 809	223
1.036	0.200 412	337	0.090 443	560	9.890 032	222
1.037	0.200 749	338	0.091 003	559	9.890 254	221
1.038	0.201 087	337	0.091 562	559	9.890 475	221
1.039	0.201 424	338	0.092 121	558	9.890 696	221
1.040	0.201 762	338	0.092 679	558	9.890 917	220
1.041	0.202 100	338	0.093 237	558	9.891 137	220
1.042	0.202 438	338	0.093 795	558	9.891 357	219
1.043	0.202 776	339	0.094 353	557	9.891 576	219
1.044	0.203 115	338	0.094 910	557	9.891 795	219
1.045	0.203 453	339	0.095 467	557	9.892 014	218
1.046	0.203 792	339	0.096 024	556	9.892 232	217
1.047	0.204 131	339	0.096 580	556	9.892 449	217
1.048	0.204 470	339	0.097 136	556	9.892 666	217
1.049	0.204 809	340	0.097 692	556	9.892 883	216

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

z	log cosh z	d	log sinh z	d	log tanh z	d
1.050	0.205 149	339	0.098 248	555	9.893 099	216
1.051	0.205 488	340	0.098 803	555	9.893 315	215
1.052	0.205 828	340	0.099 358	555	9.893 530	215
1.053	0.206 168	340	0.099 913	555	9.893 745	215
1.054	0.206 508	340	0.100 468	554	9.893 960	213
1.055	0.206 848	341	0.101 022	554	9.894 173	214
1.056	0.207 189	341	0.101 576	554	9.894 387	213
1.057	0.207 530	340	0.102 130	553	9.894 600	213
1.058	0.207 870	341	0.102 683	553	9.894 813	212
1.059	0.208 211	341	0.103 236	553	9.895 025	212
1.060	0.208 552	342	0.103 789	553	9.895 237	211
1.061	0.208 894	341	0.104 342	552	9.895 448	211
1.062	0.209 235	342	0.104 894	552	9.895 659	211
1.063	0.209 577	342	0.105 446	552	9.895 870	210
1.064	0.209 919	342	0.105 998	552	9.896 080	209
1.065	0.210 261	342	0.106 550	551	9.896 289	209
1.066	0.210 603	342	0.107 101	551	9.896 498	209
1.067	0.210 945	342	0.107 652	551	9.896 707	208
1.068	0.211 287	343	0.108 203	550	9.896 915	208
1.069	0.211 630	343	0.108 753	551	9.897 123	208
1.070	0.211 973	343	0.109 304	550	9.897 331	207
1.071	0.212 316	343	0.109 854	549	9.897 538	206
1.072	0.212 659	343	0.110 403	550	9.897 744	207
1.073	0.213 002	344	0.110 953	549	9.897 951	205
1.074	0.213 346	343	0.111 502	549	9.898 156	206
1.075	0.213 689	344	0.112 051	549	9.898 362	205
1.076	0.214 033	344	0.112 600	548	9.898 567	204
1.077	0.214 377	344	0.113 148	548	9.898 771	204
1.078	0.214 721	344	0.113 696	548	9.898 975	204
1.079	0.215 065	345	0.114 244	548	9.899 179	203
1.080	0.215 410	344	0.114 792	547	9.899 382	203
1.081	0.215 754	345	0.115 339	547	9.899 585	202
1.082	0.216 099	345	0.115 886	547	9.899 787	202
1.083	0.216 444	345	0.116 433	547	9.899 989	202
1.084	0.216 789	345	0.116 980	546	9.900 191	201
1.085	0.217 134	345	0.117 526	546	9.900 392	201
1.086	0.217 479	346	0.118 072	546	9.900 593	200
1.087	0.217 825	346	0.118 618	546	9.900 793	200
1.088	0.218 171	345	0.119 164	545	9.900 993	200
1.089	0.218 516	346	0.119 709	545	9.901 193	199
1.090	0.218 862	347	0.120 254	545	9.901 392	199
1.091	0.219 209	346	0.120 799	545	9.901 591	198
1.092	0.219 555	346	0.121 344	544	9.901 789	198
1.093	0.219 901	347	0.121 888	544	9.901 987	197
1.094	0.220 248	347	0.122 432	544	9.902 184	197
1.095	0.220 595	347	0.122 976	544	9.902 381	197
1.096	0.220 942	347	0.123 520	543	9.902 578	196
1.097	0.221 289	347	0.124 063	543	9.902 774	196
1.098	0.221 636	347	0.124 606	543	9.902 970	196
1.099	0.221 983	348	0.125 149	543	9.903 166	195

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(*Continued*)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.100	0.222 331	348	0.125 692	542	9.903 361	194
1.101	0.222 679	348	0.126 234	542	9.903 555	195
1.102	0.223 027	348	0.126 776	542	9.903 750	194
1.103	0.223 375	348	0.127 318	542	9.903 944	193
1.104	0.223 723	348	0.127 860	541	9.904 137	193
1.105	0.224 071	349	0.128 401	542	9.904 330	193
1.106	0.224 420	348	0.128 943	541	9.904 523	192
1.107	0.224 768	349	0.129 484	540	9.904 715	192
1.108	0.225 117	349	0.130 024	541	9.904 907	192
1.109	0.225 466	349	0.130 565	540	9.905 099	191
1.110	0.225 815	350	0.131 105	540	9.905 290	190
1.111	0.226 165	349	0.131 645	540	9.905 480	191
1.112	0.226 514	350	0.132 185	539	9.905 671	190
1.113	0.226 864	349	0.132 724	540	9.905 861	189
1.114	0.227 213	350	0.133 264	539	9.906 050	190
1.115	0.227 563	350	0.133 803	539	9.906 240	188
1.116	0.227 913	350	0.134 342	538	9.906 428	189
1.117	0.228 263	351	0.134 880	539	9.906 617	188
1.118	0.228 614	350	0.135 419	538	9.906 805	187
1.119	0.228 964	351	0.135 957	538	9.906 992	188
1.120	0.229 315	351	0.136 495	537	9.907 180	186
1.121	0.229 666	351	0.137 032	538	9.907 366	187
1.122	0.230 017	351	0.137 570	537	9.907 553	186
1.123	0.230 368	351	0.138 107	537	9.907 739	186
1.124	0.230 719	351	0.138 644	537	9.907 925	185
1.125	0.231 070	352	0.139 181	536	9.908 110	185
1.126	0.231 422	352	0.139 717	536	9.908 295	185
1.127	0.231 774	352	0.140 253	536	9.908 480	184
1.128	0.232 126	352	0.140 789	536	9.908 664	184
1.129	0.232 478	352	0.141 325	536	9.908 848	183
1.130	0.232 830	352	0.141 861	535	9.909 031	183
1.131	0.233 182	352	0.142 396	535	9.909 214	183
1.132	0.233 534	353	0.142 931	535	9.909 397	182
1.133	0.233 887	353	0.143 466	535	9.909 579	182
1.134	0.234 240	353	0.144 001	535	9.909 761	182
1.135	0.234 593	353	0.144 536	534	9.909 943	181
1.136	0.234 946	353	0.145 070	534	9.910 124	181
1.137	0.235 299	353	0.145 604	534	9.910 305	180
1.138	0.235 652	354	0.146 138	533	9.910 485	181
1.139	0.236 006	353	0.146 671	534	9.910 666	179
1.140	0.236 359	354	0.147 205	533	9.910 845	180
1.141	0.236 713	354	0.147 738	533	9.911 025	179
1.142	0.237 067	354	0.148 271	532	9.911 204	178
1.143	0.237 421	354	0.148 803	533	9.911 382	179
1.144	0.237 775	355	0.149 336	532	9.911 561	178
1.145	0.238 130	354	0.149 868	532	9.911 739	177
1.146	0.238 484	355	0.150 400	532	9.911 916	177
1.147	0.238 839	355	0.150 932	532	9.912 093	177
1.148	0.239 194	354	0.151 464	531	9.912 270	177
1.149	0.239 548	356	0.151 995	531	9.912 447	176

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS

(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.150	0.239 904	355	0.152 526	531	9.912 623	175
1.151	0.240 259	355	0.153 057	531	9.912 798	176
1.152	0.240 614	356	0.153 588	531	9.912 974	175
1.153	0.240 970	355	0.154 119	530	9.913 149	175
1.154	0.241 325	356	0.154 649	530	9.913 324	174
1.155	0.241 681	356	0.155 179	530	9.913 498	174
1.156	0.242 037	356	0.155 709	530	9.913 672	174
1.157	0.242 393	356	0.156 239	529	9.913 846	173
1.158	0.242 749	357	0.156 768	529	9.914 019	173
1.159	0.243 106	356	0.157 297	529	9.914 192	172
1.160	0.243 462	357	0.157 826	529	9.914 364	172
1.161	0.243 819	357	0.158 355	529	9.914 536	172
1.162	0.244 176	356	0.158 884	528	9.914 708	172
1.163	0.244 532	358	0.159 412	529	9.914 880	171
1.164	0.244 890	357	0.159 941	528	9.915 051	171
1.165	0.245 247	357	0.160 469	527	9.915 222	170
1.166	0.245 604	358	0.160 996	528	9.915 392	170
1.167	0.245 962	357	0.161 524	527	9.915 562	170
1.168	0.246 319	358	0.162 051	528	9.915 732	170
1.169	0.246 677	358	0.162 579	527	9.915 902	169
1.170	0.247 035	358	0.163 106	526	9.916 071	168
1.171	0.247 393	358	0.163 632	527	9.916 239	169
1.172	0.247 751	358	0.164 159	526	9.916 408	168
1.173	0.248 109	359	0.164 685	526	9.916 576	168
1.174	0.248 468	359	0.165 211	526	9.916 744	167
1.175	0.248 827	358	0.165 737	526	9.916 911	167
1.176	0.249 185	359	0.166 263	526	9.917 078	167
1.177	0.249 544	359	0.166 789	525	9.917 245	166
1.178	0.249 903	359	0.167 314	525	9.917 411	166
1.179	0.250 262	360	0.167 839	525	9.917 577	166
1.180	0.250 622	359	0.168 364	525	9.917 743	165
1.181	0.250 981	360	0.168 889	525	9.917 908	165
1.182	0.251 341	359	0.169 414	524	9.918 073	165
1.183	0.251 700	360	0.169 938	524	9.918 238	164
1.184	0.252 060	360	0.170 462	524	9.918 402	164
1.185	0.252 420	360	0.170 986	524	9.918 566	164
1.186	0.252 780	360	0.171 510	523	9.918 730	163
1.187	0.253 140	361	0.172 033	524	9.918 893	163
1.188	0.253 501	360	0.172 557	523	9.919 056	163
1.189	0.253 861	361	0.173 080	523	9.919 219	162
1.190	0.254 222	361	0.173 603	523	9.919 381	162
1.191	0.254 583	361	0.174 126	522	9.919 543	162
1.192	0.254 944	361	0.174 648	523	9.919 705	161
1.193	0.255 305	361	0.175 171	522	9.919 866	161
1.194	0.255 666	361	0.175 693	522	9.920 027	161
1.195	0.256 027	362	0.176 215	522	9.920 188	160
1.196	0.256 389	361	0.176 737	521	9.920 348	160
1.197	0.256 750	362	0.177 258	522	9.920 508	160
1.198	0.257 112	362	0.177 780	521	9.920 668	159
1.199	0.257 474	362	0.178 301	521	9.920 827	159

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS

(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.200	0.257 836	362	0.178 822	521	9.920 986	159
1.201	0.258 198	362	0.179 343	521	9.921 145	158
1.202	0.258 560	363	0.179 864	520	9.921 303	158
1.203	0.258 923	362	0.180 384	520	9.921 461	158
1.204	0.259 285	363	0.180 904	520	9.921 619	158
1.205	0.259 648	363	0.181 424	520	9.921 777	157
1.206	0.260 011	362	0.181 944	520	9.921 934	157
1.207	0.260 373	363	0.182 464	520	9.922 091	156
1.208	0.260 736	364	0.182 984	519	9.922 247	156
1.209	0.261 100	363	0.183 503	519	9.922 403	156
1.210	0.261 463	363	0.184 022	519	9.922 559	156
1.211	0.261 826	364	0.184 541	519	9.922 715	155
1.212	0.262 190	364	0.185 060	518	9.922 870	155
1.213	0.262 554	363	0.185 578	519	9.923 025	154
1.214	0.262 917	364	0.186 097	518	9.923 179	155
1.215	0.263 281	364	0.186 615	518	9.923 334	154
1.216	0.263 645	365	0.187 133	518	9.923 488	153
1.217	0.264 010	364	0.187 651	518	9.923 641	154
1.218	0.264 374	364	0.188 169	517	9.923 795	153
1.219	0.264 738	365	0.188 686	518	9.923 948	152
1.220	0.265 103	365	0.189 204	517	9.924 100	153
1.221	0.265 468	365	0.189 721	517	9.924 253	152
1.222	0.265 833	365	0.190 238	516	9.924 405	152
1.223	0.266 198	365	0.190 754	517	9.924 557	151
1.224	0.266 563	365	0.191 271	516	9.924 708	151
1.225	0.266 928	365	0.191 787	517	9.924 859	151
1.226	0.267 293	366	0.192 304	516	9.925 010	151
1.227	0.267 659	365	0.192 820	516	9.925 161	150
1.228	0.268 024	366	0.193 336	515	9.925 311	150
1.229	0.268 390	366	0.193 851	516	9.925 461	150
1.230	0.268 756	366	0.194 367	515	9.925 611	149
1.231	0.269 122	366	0.194 882	515	9.925 760	149
1.232	0.269 488	366	0.195 397	515	9.925 909	149
1.233	0.269 854	367	0.195 912	515	9.926 058	148
1.234	0.270 221	366	0.196 427	515	9.926 206	149
1.235	0.270 587	367	0.196 942	514	9.926 355	147
1.236	0.270 954	367	0.197 456	515	9.926 502	148
1.237	0.271 321	366	0.197 971	514	9.926 650	147
1.238	0.271 687	367	0.198 485	514	9.926 797	147
1.239	0.272 054	368	0.198 999	513	9.926 944	147
1.240	0.272 422	367	0.199 512	514	9.927 091	146
1.241	0.272 789	367	0.200 026	513	9.927 237	146
1.242	0.273 156	368	0.200 539	514	9.927 383	146
1.243	0.273 524	367	0.201 053	513	9.927 529	145
1.244	0.273 891	368	0.201 566	513	9.927 674	146
1.245	0.274 259	368	0.202 079	512	9.927 820	145
1.246	0.274 627	368	0.202 591	513	9.927 965	144
1.247	0.274 995	368	0.203 104	512	9.928 109	144
1.248	0.275 363	368	0.203 616	513	9.928 253	144
1.249	0.275 731	368	0.204 129	512	9.928 397	144

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(*Continued*)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.250	0.276 099	369	0.204 641	511	9.928 541	144
1.251	0.276 468	369	0.205 152	512	9.928 685	143
1.252	0.276 837	368	0.205 664	512	9.928 828	142
1.253	0.277 205	369	0.206 176	511	9.928 970	143
1.254	0.277 574	369	0.206 687	511	9.929 113	142
1.255	0.277 943	369	0.207 198	511	9.929 255	142
1.256	0.278 312	369	0.207 709	511	9.929 397	142
1.257	0.278 681	370	0.208 220	511	9.929 539	141
1.258	0.279 051	369	0.208 731	510	9.929 680	141
1.259	0.279 420	370	0.209 241	511	9.929 821	141
1.260	0.279 790	369	0.209 752	510	9.929 962	141
1.261	0.280 159	370	0.210 262	510	9.930 103	140
1.262	0.280 529	370	0.210 772	510	9.930 243	140
1.263	0.280 899	370	0.211 282	510	9.930 383	140
1.264	0.281 269	370	0.211 792	509	9.930 523	139
1.265	0.281 639	370	0.212 301	510	9.930 662	139
1.266	0.282 009	371	0.212 811	509	9.930 801	139
1.267	0.282 380	370	0.213 320	509	9.930 940	139
1.268	0.282 750	371	0.213 829	509	9.931 079	138
1.269	0.283 121	371	0.214 338	509	9.931 217	138
1.270	0.283 492	371	0.214 847	508	9.931 355	138
1.271	0.283 863	370	0.215 355	509	9.931 493	137
1.272	0.284 233	372	0.215 864	508	9.931 630	137
1.273	0.284 605	371	0.216 372	508	9.931 767	137
1.274	0.284 976	371	0.216 880	508	9.931 904	137
1.275	0.285 347	372	0.217 388	508	9.932 041	136
1.276	0.285 719	371	0.217 896	507	9.932 177	136
1.277	0.286 090	372	0.218 403	508	9.932 313	136
1.278	0.286 462	372	0.218 911	507	9.932 449	135
1.279	0.286 834	372	0.219 418	507	9.932 584	136
1.280	0.287 206	372	0.219 925	507	9.932 720	135
1.281	0.287 578	372	0.220 432	507	9.932 855	134
1.282	0.287 950	372	0.220 939	507	9.932 989	135
1.283	0.288 322	372	0.221 446	506	9.933 124	134
1.284	0.288 694	373	0.221 952	507	9.933 258	134
1.285	0.289 067	372	0.222 459	506	9.933 392	133
1.286	0.289 439	373	0.222 965	506	9.933 525	134
1.287	0.289 812	373	0.223 471	506	9.933 659	133
1.288	0.290 185	373	0.223 977	505	9.933 792	133
1.289	0.290 558	373	0.224 482	506	9.933 925	132
1.290	0.290 931	373	0.224 988	505	9.934 057	132
1.291	0.291 304	373	0.225 493	506	9.934 189	132
1.292	0.291 677	374	0.225 999	505	9.934 321	132
1.293	0.292 051	373	0.226 504	505	9.934 453	132
1.294	0.292 424	374	0.227 009	505	9.934 585	131
1.295	0.292 798	374	0.227 514	504	9.934 716	131
1.296	0.293 172	374	0.228 018	505	9.934 847	130
1.297	0.293 546	373	0.228 523	504	9.934 977	131
1.298	0.293 919	375	0.229 027	504	9.935 108	130
1.299	0.294 294	374	0.229 531	504	9.935 238	130

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS

(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.300	0.294 668	374	0.230 035	504	9.935 368	129
1.301	0.295 042	374	0.230 539	504	9.935 497	130
1.302	0.295 416	375	0.231 043	504	9.935 627	129
1.303	0.295 791	375	0.231 547	503	9.935 756	129
1.304	0.296 166	374	0.232 050	504	9.935 885	128
1.305	0.296 540	375	0.232 554	503	9.936 013	129
1.306	0.296 915	375	0.233 057	503	9.936 142	128
1.307	0.297 290	375	0.233 560	503	9.936 270	127
1.308	0.297 665	375	0.234 063	502	9.936 397	128
1.309	0.298 040	376	0.234 565	503	9.936 525	127
1.310	0.298 416	375	0.235 068	502	9.936 652	127
1.311	0.298 791	376	0.235 570	503	9.936 779	127
1.312	0.299 167	375	0.236 073	502	9.936 906	126
1.313	0.299 542	376	0.236 575	502	9.937 032	127
1.314	0.299 918	376	0.237 077	502	9.937 159	126
1.315	0.300 294	376	0.237 579	501	9.937 285	125
1.316	0.300 670	376	0.238 080	502	9.937 410	126
1.317	0.301 046	376	0.238 582	501	9.937 536	125
1.318	0.301 422	376	0.239 083	501	9.937 661	125
1.319	0.301 798	377	0.239 584	502	9.937 786	125
1.320	0.302 175	376	0.240 086	500	9.937 911	124
1.321	0.302 551	377	0.240 586	501	9.938 035	125
1.322	0.302 928	376	0.241 087	501	9.938 160	124
1.323	0.303 304	377	0.241 588	501	9.938 284	123
1.324	0.303 681	377	0.242 089	500	9.938 407	124
1.325	0.304 058	377	0.242 589	500	9.938 531	123
1.326	0.304 435	377	0.243 089	500	9.938 654	123
1.327	0.304 812	378	0.243 589	500	9.938 777	123
1.328	0.305 190	377	0.244 089	500	9.938 900	122
1.329	0.305 567	377	0.244 589	500	9.939 022	122
1.330	0.305 944	378	0.245 089	499	9.939 144	122
1.331	0.306 322	378	0.245 588	500	9.939 266	122
1.332	0.306 700	377	0.246 088	499	9.939 388	122
1.333	0.307 077	378	0.246 587	499	9.939 510	121
1.334	0.307 455	378	0.247 086	499	9.939 631	121
1.335	0.307 833	378	0.247 585	499	9.939 752	121
1.336	0.308 211	379	0.248 084	499	9.939 873	120
1.337	0.308 590	378	0.248 583	498	9.939 993	120
1.338	0.308 968	378	0.249 081	499	9.940 113	120
1.339	0.309 346	379	0.249 580	498	9.940 233	120
1.340	0.309 725	378	0.250 078	498	9.940 353	120
1.341	0.310 103	379	0.250 576	498	9.940 473	119
1.342	0.310 482	379	0.251 074	498	9.940 592	119
1.343	0.310 861	379	0.251 572	498	9.940 711	119
1.344	0.311 240	379	0.252 070	497	9.940 830	119
1.345	0.311 619	379	0.252 567	498	9.940 949	118
1.346	0.311 998	379	0.253 065	497	9.941 067	118
1.347	0.312 377	380	0.253 562	498	9.941 185	118
1.348	0.312 757	379	0.254 060	497	9.941 303	118
1.349	0.313 136	380	0.254 557	497	9.941 421	117

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.350	0.313 516	379	0.255 054	496	9.941 538	117
1.351	0.313 895	380	0.255 550	497	9.941 655	117
1.352	0.314 275	380	0.256 047	497	9.941 772	117
1.353	0.314 655	380	0.256 544	496	9.941 889	116
1.354	0.315 035	380	0.257 040	496	9.942 005	116
1.355	0.315 415	380	0.257 536	496	9.942 121	116
1.356	0.315 795	380	0.258 032	496	9.942 237	116
1.357	0.316 175	381	0.258 528	496	9.942 353	116
1.358	0.316 556	380	0.259 024	496	9.942 469	115
1.359	0.316 936	381	0.259 520	496	9.942 584	115
1.360	0.317 317	380	0.260 016	495	9.942 699	115
1.361	0.317 697	381	0.260 511	495	9.942 814	114
1.362	0.318 078	381	0.261 006	496	9.942 928	115
1.363	0.318 459	381	0.261 502	495	9.943 043	114
1.364	0.318 840	381	0.261 997	495	9.943 157	114
1.365	0.319 221	381	0.262 492	495	9.943 271	113
1.366	0.319 602	381	0.262 987	494	9.943 384	114
1.367	0.319 983	382	0.263 481	495	9.943 498	113
1.368	0.320 365	381	0.263 976	494	9.943 611	113
1.369	0.320 746	382	0.264 470	495	9.943 724	113
1.370	0.321 128	381	0.264 965	494	9.943 837	112
1.371	0.321 509	382	0.265 459	494	9.943 949	113
1.372	0.321 891	382	0.265 953	494	9.944 062	112
1.373	0.322 273	382	0.266 447	494	9.944 174	112
1.374	0.322 655	382	0.266 941	493	9.944 286	111
1.375	0.323 037	382	0.267 434	494	9.944 397	112
1.376	0.323 419	383	0.267 928	493	9.944 509	111
1.377	0.323 802	382	0.268 421	494	9.944 620	111
1.378	0.324 184	382	0.268 915	493	9.944 731	110
1.379	0.324 566	383	0.269 408	493	9.944 841	111
1.380	0.324 949	382	0.269 901	493	9.944 952	110
1.381	0.325 331	383	0.270 394	492	9.945 062	110
1.382	0.325 714	383	0.270 886	493	9.945 172	110
1.383	0.326 097	383	0.271 379	493	9.945 282	110
1.384	0.326 480	383	0.271 872	492	9.945 392	109
1.385	0.326 863	383	0.272 364	492	9.945 501	109
1.386	0.327 246	383	0.272 856	493	9.945 610	109
1.387	0.327 629	384	0.273 349	492	9.945 719	109
1.388	0.328 013	383	0.273 841	492	9.945 828	108
1.389	0.328 396	384	0.274 333	491	9.945 936	109
1.390	0.328 780	383	0.274 824	492	9.946 045	108
1.391	0.329 163	384	0.275 316	492	9.946 153	108
1.392	0.329 547	384	0.275 808	491	9.946 261	107
1.393	0.329 931	384	0.276 299	491	9.946 368	108
1.394	0.330 315	384	0.276 790	492	9.946 476	107
1.395	0.330 699	384	0.277 282	491	9.946 583	107
1.396	0.331 083	384	0.277 773	491	9.946 690	107
1.397	0.331 467	384	0.278 264	490	9.946 797	106
1.398	0.331 851	384	0.278 754	491	9.946 903	107
1.399	0.332 235	385	0.279 245	491	9.947 010	106

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS

(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.400	0.332 620	384	0.279 736	490	9.947 116	106
1.401	0.333 004	385	0.280 226	491	9.947 222	105
1.402	0.333 389	385	0.280 717	490	9.947 327	106
1.403	0.333 774	385	0.281 207	490	9.947 433	105
1.404	0.334 159	385	0.281 697	490	9.947 538	105
1.405	0.334 544	385	0.282 187	490	9.947 643	105
1.406	0.334 929	385	0.282 677	490	9.947 748	105
1.407	0.335 314	385	0.283 167	489	9.947 853	104
1.408	0.335 699	385	0.283 656	490	9.947 957	104
1.409	0.336 084	386	0.284 146	489	9.948 061	104
1.410	0.336 470	385	0.284 635	489	9.948 165	104
1.411	0.336 855	386	0.285 124	490	9.948 269	104
1.412	0.337 241	385	0.285 614	489	9.948 373	103
1.413	0.337 626	386	0.286 103	489	9.948 476	103
1.414	0.338 012	386	0.286 592	488	9.948 579	103
1.415	0.338 398	386	0.287 080	489	9.948 682	103
1.416	0.338 784	386	0.287 569	489	9.948 785	103
1.417	0.339 170	386	0.288 058	488	9.948 888	102
1.418	0.339 556	386	0.288 546	489	9.948 990	102
1.419	0.339 942	387	0.289 035	488	9.949 092	102
1.420	0.340 329	386	0.289 523	488	9.949 194	102
1.421	0.340 715	386	0.290 011	488	9.949 296	102
1.422	0.341 101	387	0.290 499	488	9.949 398	101
1.423	0.341 488	387	0.290 987	488	9.949 499	101
1.424	0.341 875	386	0.291 475	487	9.949 600	101
1.425	0.342 261	387	0.291 962	488	9.949 701	101
1.426	0.342 648	387	0.292 450	487	9.949 802	100
1.427	0.343 035	387	0.292 937	488	9.949 902	100
1.428	0.343 422	387	0.293 425	487	9.950 002	101
1.429	0.343 809	388	0.293 912	487	9.950 103	99
1.430	0.344 197	387	0.294 399	487	9.950 202	100
1.431	0.344 584	387	0.294 886	487	9.950 302	100
1.432	0.344 971	388	0.295 373	487	9.950 402	99
1.433	0.345 359	387	0.295 860	486	9.950 501	99
1.434	0.345 746	388	0.296 346	487	9.950 600	99
1.435	0.346 134	388	0.296 833	486	9.950 699	99
1.436	0.346 522	387	0.297 319	487	9.950 798	98
1.437	0.346 909	388	0.297 806	486	9.950 896	99
1.438	0.347 297	388	0.298 292	486	9.950 995	98
1.439	0.347 685	388	0.298 778	486	9.951 093	98
1.440	0.348 073	389	0.299 264	486	9.951 191	97
1.441	0.348 462	388	0.299 750	486	9.951 288	98
1.442	0.348 850	388	0.300 236	485	9.951 386	97
1.443	0.349 238	389	0.300 721	486	9.951 483	97
1.444	0.349 627	388	0.301 207	485	9.951 580	97
1.445	0.350 015	389	0.301 692	486	9.951 677	97
1.446	0.350 404	388	0.302 178	485	9.951 774	97
1.447	0.350 792	389	0.302 663	485	9.951 871	96
1.448	0.351 181	389	0.303 148	485	9.951 967	96
1.449	0.351 570	389	0.303 633	485	9.952 063	96

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.450	0.351 959	389	0.304 118	485	9.952 159	96
1.451	0.352 348	389	0.304 603	485	9.952 255	95
1.452	0.352 737	389	0.305 088	484	9.952 350	96
1.453	0.353 126	390	0.305 572	485	9.952 446	95
1.454	0.353 516	389	0.306 057	484	9.952 541	95
1.455	0.353 905	390	0.306 541	484	9.952 636	95
1.456	0.354 295	389	0.307 025	485	9.952 731	95
1.457	0.354 684	390	0.307 510	484	9.952 826	94
1.458	0.355 074	389	0.307 994	484	9.952 920	94
1.459	0.355 463	390	0.308 478	484	9.953 014	94
1.460	0.355 853	390	0.308 962	483	9.953 108	94
1.461	0.356 243	390	0.309 445	484	9.953 202	94
1.462	0.356 633	390	0.309 929	483	9.953 296	93
1.463	0.357 023	390	0.310 412	484	9.953 389	94
1.464	0.357 413	391	0.310 896	483	9.953 483	93
1.465	0.357 804	390	0.311 379	484	9.953 576	93
1.466	0.358 194	390	0.311 863	483	9.953 669	92
1.467	0.358 584	391	0.312 346	483	9.953 761	93
1.468	0.358 975	390	0.312 829	483	9.953 854	92
1.469	0.359 365	391	0.313 312	482	9.953 946	93
1.470	0.359 756	391	0.313 794	483	9.954 039	92
1.471	0.360 147	390	0.314 277	483	9.954 131	91
1.472	0.360 537	391	0.314 760	482	9.954 222	92
1.473	0.360 928	391	0.315 242	483	9.954 314	91
1.474	0.361 319	391	0.315 725	482	9.954 405	92
1.475	0.361 710	391	0.316 207	482	9.954 497	91
1.476	0.362 101	392	0.316 689	482	9.954 588	91
1.477	0.362 493	391	0.317 171	482	9.954 679	90
1.478	0.362 884	391	0.317 653	482	9.954 769	91
1.479	0.363 275	392	0.318 135	482	9.954 860	90
1.480	0.363 667	391	0.318 617	482	9.954 950	90
1.481	0.364 058	392	0.319 099	481	9.955 040	90
1.482	0.364 450	392	0.319 580	482	9.955 130	90
1.483	0.364 842	391	0.320 062	481	9.955 220	90
1.484	0.365 233	392	0.320 543	482	9.955 310	89
1.485	0.365 625	392	0.321 025	481	9.955 399	90
1.486	0.366 017	392	0.321 506	481	9.955 489	89
1.487	0.366 409	392	0.321 987	481	9.955 578	89
1.488	0.366 801	393	0.322 468	481	9.955 667	88
1.489	0.367 194	392	0.322 949	481	9.955 755	89
1.490	0.367 586	392	0.323 430	481	9.955 844	88
1.491	0.367 978	393	0.323 911	480	9.955 932	88
1.492	0.368 371	392	0.324 391	481	9.956 020	89
1.493	0.368 763	393	0.324 872	480	9.956 109	87
1.494	0.369 156	392	0.325 352	480	9.956 196	88
1.495	0.369 548	393	0.325 832	481	9.956 284	88
1.496	0.369 941	393	0.326 313	480	9.956 372	87
1.497	0.370 334	393	0.326 793	480	9.956 459	87
1.498	0.370 727	393	0.327 273	480	9.956 546	87
1.499	0.371 120	393	0.327 753	480	9.956 633	87

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.500	0.371 513	393	0.328 233	479	9.956 720	86
1.501	0.371 906	393	0.328 712	480	9.956 806	87
1.502	0.372 299	394	0.329 192	480	9.956 893	86
1.503	0.372 693	393	0.329 672	479	9.956 979	86
1.504	0.373 086	393	0.330 151	479	9.957 065	86
1.505	0.373 479	394	0.330 630	480	9.957 151	86
1.506	0.373 873	394	0.331 110	479	9.957 237	85
1.507	0.374 267	393	0.331 589	479	9.957 322	86
1.508	0.374 660	394	0.332 068	479	9.957 408	85
1.509	0.375 054	394	0.332 547	479	9.957 493	85
1.510	0.375 448	394	0.333 026	479	9.957 578	85
1.511	0.375 842	394	0.333 505	478	9.957 663	85
1.512	0.376 236	394	0.333 983	479	9.957 748	84
1.513	0.376 630	394	0.334 462	479	9.957 832	85
1.514	0.377 024	394	0.334 941	478	9.957 917	84
1.515	0.377 418	395	0.335 419	478	9.958 001	84
1.516	0.377 813	394	0.335 897	479	9.958 085	84
1.517	0.378 207	394	0.336 376	478	9.958 169	83
1.518	0.378 601	395	0.336 854	478	9.958 252	84
1.519	0.378 996	394	0.337 332	478	9.958 336	83
1.520	0.379 390	395	0.337 810	478	9.958 419	84
1.521	0.379 785	395	0.338 288	478	9.958 503	83
1.522	0.380 180	395	0.338 766	477	9.958 586	83
1.523	0.380 575	395	0.339 243	478	9.958 669	82
1.524	0.380 970	395	0.339 721	477	9.958 751	83
1.525	0.381 365	395	0.340 198	478	9.958 834	82
1.526	0.381 760	395	0.340 676	477	9.958 916	82
1.527	0.382 155	395	0.341 153	477	9.958 998	82
1.528	0.382 550	395	0.341 630	478	9.959 080	82
1.529	0.382 945	396	0.342 108	477	9.959 162	82
1.530	0.383 341	395	0.342 585	477	9.959 244	82
1.531	0.383 736	396	0.343 062	477	9.959 326	81
1.532	0.384 132	395	0.343 539	476	9.959 407	81
1.533	0.384 527	396	0.344 015	477	9.959 488	81
1.534	0.384 923	396	0.344 492	477	9.959 569	81
1.535	0.385 319	395	0.344 969	476	9.959 650	81
1.536	0.385 714	396	0.345 445	477	9.959 731	80
1.537	0.386 110	396	0.345 922	476	9.959 811	81
1.538	0.386 506	396	0.346 398	476	9.959 892	80
1.539	0.386 902	396	0.346 874	476	9.959 972	80
1.540	0.387 298	396	0.347 350	477	9.960 052	80
1.541	0.387 694	397	0.347 827	476	9.960 132	80
1.542	0.388 091	396	0.348 303	475	9.960 212	79
1.543	0.388 487	396	0.348 778	476	9.960 291	80
1.544	0.388 883	397	0.349 254	476	9.960 371	79
1.545	0.389 280	396	0.349 730	476	9.960 450	79
1.546	0.389 676	397	0.350 206	475	9.960 529	79
1.547	0.390 073	397	0.350 681	476	9.960 608	79
1.548	0.390 470	396	0.351 157	475	9.960 687	79
1.549	0.390 866	397	0.351 632	475	9.960 766	78

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(*Continued*)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.550	0.391 263	397	0.352 107	476	9.960 844	79
1.551	0.391 660	397	0.352 583	475	9.960 923	78
1.552	0.392 057	397	0.353 058	475	9.961 001	78
1.553	0.392 454	397	0.353 533	475	9.961 079	78
1.554	0.392 851	397	0.354 008	475	9.961 157	77
1.555	0.393 248	398	0.354 483	475	9.961 234	78
1.556	0.393 646	397	0.354 958	474	9.961 312	77
1.557	0.394 043	397	0.355 432	475	9.961 389	78
1.558	0.394 440	398	0.355 907	474	9.961 467	77
1.559	0.394 838	397	0.356 381	475	9.961 544	77
1.560	0.395 235	398	0.356 856	474	9.961 621	76
1.561	0.395 633	398	0.357 330	475	9.961 697	77
1.562	0.396 031	397	0.357 805	474	9.961 774	76
1.563	0.396 428	398	0.358 279	474	9.961 850	77
1.564	0.396 826	398	0.358 753	474	9.961 927	76
1.565	0.397 224	398	0.359 227	474	9.962 003	76
1.566	0.397 622	398	0.359 701	474	9.962 079	76
1.567	0.398 020	398	0.360 175	474	9.962 155	76
1.568	0.398 418	398	0.360 649	473	9.962 231	75
1.569	0.398 816	398	0.361 122	474	9.962 306	76
1.570	0.399 214	399	0.361 596	473	9.962 382	75
1.571	0.399 613	398	0.362 069	474	9.962 457	75
1.572	0.400 011	398	0.362 543	473	9.962 532	75
1.573	0.400 409	399	0.363 016	474	9.962 607	75
1.574	0.400 808	399	0.363 490	473	9.962 682	74
1.575	0.401 207	398	0.363 963	473	9.962 756	75
1.576	0.401 605	399	0.364 436	473	9.962 831	74
1.577	0.402 004	399	0.364 909	473	9.962 905	74
1.578	0.402 403	399	0.365 382	473	9.962 979	74
1.579	0.402 802	398	0.365 855	473	9.963 053	74
1.580	0.403 200	399	0.366 328	472	9.963 127	74
1.581	0.403 599	399	0.366 800	473	9.963 201	74
1.582	0.403 998	400	0.367 273	473	9.963 275	73
1.583	0.404 398	399	0.367 746	472	9.963 348	73
1.584	0.404 797	399	0.368 218	473	9.963 421	74
1.585	0.405 196	399	0.368 691	472	9.963 495	73
1.586	0.405 595	400	0.369 163	472	9.963 568	73
1.587	0.405 995	399	0.369 635	472	9.963 641	72
1.588	0.406 394	400	0.370 107	473	9.963 713	73
1.589	0.406 794	399	0.370 580	472	9.963 786	72
1.590	0.407 193	400	0.371 052	471	9.963 858	73
1.591	0.407 593	400	0.371 523	472	9.963 931	72
1.592	0.407 993	399	0.371 995	472	9.964 003	72
1.593	0.408 392	400	0.372 467	472	9.964 075	72
1.594	0.408 792	400	0.372 939	471	9.964 147	71
1.595	0.409 192	400	0.373 410	472	9.964 218	72
1.596	0.409 592	400	0.373 882	471	9.964 290	71
1.597	0.409 992	400	0.374 353	472	9.964 361	72
1.598	0.410 392	400	0.374 825	471	9.964 433	71
1.599	0.410 792	401	0.375 296	471	9.964 504	71

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS

(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.600	0.411 193	400	0.375 767	472	9.964 575	71
1.601	0.411 593	400	0.376 239	471	9.964 646	70
1.602	0.411 993	401	0.376 710	471	9.964 716	71
1.603	0.412 394	400	0.377 181	471	9.964 787	70
1.604	0.412 794	401	0.377 652	471	9.964 857	71
1.605	0.413 195	401	0.378 123	470	9.964 928	70
1.606	0.413 596	400	0.378 593	471	9.964 998	70
1.607	0.413 996	401	0.379 064	471	9.965 068	70
1.608	0.414 397	401	0.379 535	470	9.965 138	69
1.609	0.414 798	401	0.380 005	471	9.965 207	70
1.610	0.415 199	401	0.380 476	470	9.965 277	69
1.611	0.415 600	401	0.380 946	470	9.965 346	70
1.612	0.416 001	401	0.381 416	471	9.965 416	69
1.613	0.416 402	401	0.381 887	470	9.965 485	69
1.614	0.416 803	401	0.382 357	470	9.965 554	69
1.615	0.417 204	401	0.382 827	470	9.965 623	69
1.616	0.417 605	402	0.383 297	470	9.965 692	68
1.617	0.418 007	401	0.383 767	470	9.965 760	69
1.618	0.418 408	402	0.384 237	470	9.965 829	68
1.619	0.418 810	401	0.384 707	469	9.965 897	68
1.620	0.419 211	402	0.385 176	470	9.965 965	68
1.621	0.419 613	401	0.385 646	470	9.966 033	68
1.622	0.420 014	402	0.386 116	469	9.966 101	68
1.623	0.420 416	402	0.386 585	470	9.966 169	68
1.624	0.420 818	402	0.387 055	469	9.966 237	67
1.625	0.421 220	402	0.387 524	469	9.966 304	68
1.626	0.421 622	402	0.387 993	469	9.966 372	67
1.627	0.422 024	402	0.388 462	470	9.966 439	67
1.628	0.422 426	402	0.388 932	469	9.966 506	67
1.629	0.422 828	402	0.389 401	469	9.966 573	67
1.630	0.423 230	402	0.389 870	469	9.966 640	67
1.631	0.423 632	402	0.390 339	468	9.966 707	66
1.632	0.424 034	403	0.390 807	469	9.966 773	67
1.633	0.424 437	402	0.391 276	469	9.966 840	66
1.634	0.424 839	403	0.391 745	469	9.966 906	66
1.635	0.425 242	402	0.392 214	468	9.966 972	66
1.636	0.425 644	403	0.392 682	469	9.967 038	66
1.637	0.426 047	402	0.393 151	468	9.967 104	66
1.638	0.426 449	403	0.393 619	469	9.967 170	66
1.639	0.426 852	403	0.394 088	468	9.967 236	65
1.640	0.427 255	403	0.394 556	468	9.967 301	65
1.641	0.427 658	402	0.395 024	468	9.967 366	66
1.642	0.428 060	403	0.395 492	468	9.967 432	65
1.643	0.428 463	403	0.395 960	468	9.967 497	65
1.644	0.428 866	403	0.396 428	468	9.967 562	65
1.645	0.429 269	404	0.396 896	468	9.967 627	64
1.646	0.429 673	403	0.397 364	468	9.967 691	65
1.647	0.430 076	403	0.397 832	468	9.967 756	65
1.648	0.430 479	403	0.398 300	467	9.967 821	64
1.649	0.430 882	404	0.398 767	468	9.967 885	64

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.650	0.431 286	403	0.399 235	467	9.967 949	64
1.651	0.431 689	404	0.399 702	468	9.968 013	64
1.652	0.432 093	403	0.400 170	467	9.968 077	64
1.653	0.432 496	404	0.400 637	468	9.968 141	64
1.654	0.432 900	403	0.401 105	467	9.968 205	63
1.655	0.433 303	404	0.401 572	467	9.968 268	64
1.656	0.433 707	404	0.402 039	467	9.968 332	63
1.657	0.434 111	404	0.402 506	467	9.968 395	63
1.658	0.434 515	404	0.402 973	467	9.968 458	63
1.659	0.434 919	404	0.403 440	467	9.968 521	63
1.660	0.435 323	404	0.403 907	467	9.968 584	63
1.661	0.435 727	404	0.404 374	467	9.968 647	63
1.662	0.436 131	404	0.404 841	466	9.968 710	62
1.663	0.436 535	404	0.405 307	467	9.968 772	63
1.664	0.436 939	404	0.405 774	467	9.968 835	62
1.665	0.437 343	405	0.406 241	466	9.968 897	62
1.666	0.437 748	404	0.406 707	466	9.968 959	62
1.667	0.438 152	404	0.407 173	467	9.969 021	62
1.668	0.438 556	405	0.407 640	466	9.969 083	62
1.669	0.438 961	404	0.408 106	466	9.969 145	62
1.670	0.439 365	405	0.408 572	467	9.969 207	62
1.671	0.439 770	405	0.409 039	466	9.969 269	61
1.672	0.440 175	404	0.409 505	466	9.969 330	61
1.673	0.440 579	405	0.409 971	466	9.969 391	62
1.674	0.440 984	405	0.410 437	466	9.969 453	61
1.675	0.441 389	405	0.410 903	465	9.969 514	61
1.676	0.441 794	405	0.411 368	466	9.969 575	60
1.677	0.442 199	405	0.411 834	466	9.969 635	61
1.678	0.442 604	405	0.412 300	466	9.969 696	61
1.679	0.443 009	405	0.412 766	465	9.969 757	60
1.680	0.443 414	405	0.413 231	466	9.969 817	61
1.681	0.443 819	405	0.413 697	465	9.969 878	60
1.682	0.444 224	406	0.414 162	466	9.969 938	60
1.683	0.444 630	405	0.414 628	465	9.969 998	60
1.684	0.445 035	405	0.415 093	465	9.970 058	60
1.685	0.445 440	406	0.415 558	465	9.970 118	60
1.686	0.445 846	405	0.416 023	465	9.970 178	59
1.687	0.446 251	406	0.416 488	466	9.970 237	60
1.688	0.446 657	405	0.416 954	465	9.970 297	59
1.689	0.447 062	406	0.417 419	464	9.970 356	59
1.690	0.447 468	406	0.417 883	465	9.970 415	59
1.691	0.447 874	406	0.418 348	465	9.970 474	60
1.692	0.448 280	405	0.418 813	465	9.970 534	58
1.693	0.448 685	406	0.419 278	465	9.970 592	59
1.694	0.449 091	406	0.419 743	464	9.970 651	59
1.695	0.449 497	406	0.420 207	465	9.970 710	58
1.696	0.449 903	406	0.420 672	464	9.970 768	59
1.697	0.450 309	406	0.421 136	465	9.970 827	58
1.698	0.450 715	407	0.421 601	464	9.970 885	58
1.699	0.451 122	406	0.422 065	464	9.970 943	59

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.700	0.451 528	406	0.422 529	465	9.971 002	58
1.701	0.451 934	406	0.422 994	464	9.971 060	57
1.702	0.452 340	407	0.423 458	464	9.971 117	58
1.703	0.452 747	406	0.423 922	464	9.971 175	58
1.704	0.453 153	407	0.424 386	464	9.971 233	57
1.705	0.453 560	406	0.424 850	464	9.971 290	58
1.706	0.453 966	407	0.425 314	464	9.971 348	57
1.707	0.454 373	407	0.425 778	464	9.971 405	57
1.708	0.454 780	406	0.426 242	463	9.971 462	57
1.709	0.455 186	407	0.426 705	464	9.971 519	57
1.710	0.455 593	407	0.427 169	464	9.971 576	57
1.711	0.456 000	407	0.427 633	463	9.971 633	57
1.712	0.456 407	407	0.428 096	464	9.971 690	56
1.713	0.456 814	407	0.428 560	463	9.971 746	57
1.714	0.457 221	407	0.429 023	464	9.971 803	56
1.715	0.457 628	407	0.429 487	463	9.971 859	56
1.716	0.458 035	407	0.429 950	463	9.971 915	57
1.717	0.458 442	407	0.430 413	464	9.971 972	56
1.718	0.458 849	407	0.430 877	463	9.972 028	56
1.719	0.459 256	407	0.431 340	463	9.972 084	55
1.720	0.459 663	408	0.431 803	463	9.972 139	56
1.721	0.460 071	407	0.432 266	463	9.972 195	56
1.722	0.460 478	408	0.432 729	463	9.972 251	55
1.723	0.460 886	407	0.433 192	463	9.972 306	56
1.724	0.461 293	408	0.433 655	462	9.972 362	55
1.725	0.461 701	407	0.434 117	463	9.972 417	55
1.726	0.462 108	408	0.434 580	463	9.972 472	55
1.727	0.462 516	408	0.435 043	463	9.972 527	55
1.728	0.462 924	407	0.435 506	462	9.972 582	55
1.729	0.463 331	408	0.435 968	463	9.972 637	54
1.730	0.463 739	408	0.436 431	462	9.972 691	55
1.731	0.464 147	408	0.436 893	462	9.972 746	55
1.732	0.464 555	408	0.437 355	463	9.972 801	54
1.733	0.464 963	408	0.437 818	462	9.972 855	54
1.734	0.465 371	408	0.438 280	462	9.972 909	54
1.735	0.465 779	408	0.438 742	462	9.972 963	54
1.736	0.466 187	408	0.439 204	463	9.973 017	54
1.737	0.466 595	408	0.439 667	462	9.973 071	54
1.738	0.467 003	409	0.440 129	462	9.973 125	54
1.739	0.467 412	408	0.440 591	461	9.973 179	54
1.740	0.467 820	408	0.441 052	462	9.973 233	53
1.741	0.468 228	409	0.441 514	462	9.973 286	53
1.742	0.468 637	408	0.441 976	462	9.973 339	54
1.743	0.469 045	409	0.442 438	462	9.973 393	53
1.744	0.469 454	408	0.442 900	461	9.973 446	53
1.745	0.469 862	409	0.443 361	462	9.973 499	53
1.746	0.470 271	409	0.443 823	461	9.973 552	53
1.747	0.470 680	408	0.444 284	462	9.973 605	53
1.748	0.471 088	409	0.444 746	461	9.973 658	52
1.749	0.471 497	409	0.445 207	462	9.973 710	53

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.750	0.471 906	409	0.445 669	461	9.973 763	52
1.751	0.472 315	409	0.446 130	461	9.973 815	53
1.752	0.472 724	409	0.446 591	461	9.973 868	52
1.753	0.473 133	409	0.447 052	462	9.973 920	52
1.754	0.473 542	409	0.447 514	461	9.973 972	52
1.755	0.473 951	409	0.447 975	461	9.974 024	52
1.756	0.474 360	409	0.448 436	461	9.974 076	52
1.757	0.474 769	409	0.448 897	461	9.974 128	52
1.758	0.475 178	409	0.449 358	461	9.974 180	51
1.759	0.475 587	410	0.449 819	460	9.974 231	52
1.760	0.475 997	409	0.450 279	461	9.974 283	51
1.761	0.476 406	409	0.450 740	461	9.974 334	51
1.762	0.476 815	410	0.451 201	460	9.974 385	52
1.763	0.477 225	409	0.451 661	461	9.974 437	51
1.764	0.477 634	410	0.452 122	461	9.974 488	51
1.765	0.478 044	409	0.452 583	460	9.974 539	51
1.766	0.478 453	410	0.453 043	461	9.974 590	50
1.767	0.478 863	410	0.453 504	460	9.974 640	51
1.768	0.479 273	409	0.453 964	460	9.974 691	51
1.769	0.479 682	410	0.454 424	461	9.974 742	50
1.770	0.480 092	410	0.454 885	460	9.974 792	51
1.771	0.480 502	410	0.455 345	460	9.974 843	50
1.772	0.480 912	410	0.455 805	460	9.974 893	50
1.773	0.481 322	410	0.456 265	460	9.974 943	50
1.774	0.481 732	410	0.456 725	460	9.974 993	50
1.775	0.482 142	410	0.457 185	460	9.975 043	50
1.776	0.482 552	410	0.457 645	460	9.975 093	50
1.777	0.482 962	410	0.458 105	460	9.975 143	50
1.778	0.483 372	410	0.458 565	460	9.975 193	49
1.779	0.483 782	411	0.459 025	459	9.975 242	50
1.780	0.484 193	410	0.459 484	460	9.975 292	49
1.781	0.484 603	410	0.459 944	460	9.975 341	49
1.782	0.485 013	411	0.460 404	459	9.975 390	50
1.783	0.485 424	410	0.460 863	460	9.975 440	49
1.784	0.485 834	411	0.461 323	459	9.975 489	49
1.785	0.486 245	410	0.461 782	460	9.975 538	49
1.786	0.486 655	411	0.462 242	459	9.975 587	48
1.787	0.487 066	410	0.462 701	459	9.975 635	49
1.788	0.487 476	411	0.463 160	460	9.975 684	49
1.789	0.487 887	411	0.463 620	459	9.975 733	48
1.790	0.488 298	410	0.464 079	459	9.975 781	49
1.791	0.488 708	411	0.464 538	459	9.975 830	48
1.792	0.489 119	411	0.464 997	459	9.975 878	48
1.793	0.489 530	411	0.465 456	459	9.975 926	48
1.794	0.489 941	411	0.465 915	459	9.975 974	48
1.795	0.490 352	411	0.466 374	459	9.976 022	48
1.796	0.490 763	411	0.466 833	459	9.976 070	48
1.797	0.491 174	411	0.467 292	459	9.976 118	48
1.798	0.491 585	411	0.467 751	459	9.976 166	47
1.799	0.491 996	411	0.468 210	458	9.976 213	48

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.800	0.492 407	412	0.468 668	459	9.976 261	47
1.801	0.492 819	411	0.469 127	459	9.976 308	48
1.802	0.493 230	411	0.469 586	458	9.976 356	47
1.803	0.493 641	412	0.470 044	459	9.976 403	47
1.804	0.494 053	411	0.470 503	458	9.976 450	47
1.805	0.494 464	411	0.470 961	459	9.976 497	47
1.806	0.494 875	412	0.471 420	458	9.976 544	47
1.807	0.495 287	411	0.471 878	458	9.976 591	47
1.808	0.495 698	412	0.472 336	459	9.976 638	47
1.809	0.496 110	412	0.472 795	458	9.976 685	46
1.810	0.496 522	411	0.473 253	458	9.976 731	47
1.811	0.496 933	412	0.473 711	458	9.976 778	46
1.812	0.497 345	412	0.474 169	458	9.976 824	47
1.813	0.497 757	411	0.474 627	458	9.976 871	46
1.814	0.498 168	412	0.475 085	458	9.976 917	46
1.815	0.498 580	412	0.475 543	458	9.976 963	46
1.816	0.498 992	412	0.476 001	458	9.977 009	46
1.817	0.499 404	412	0.476 459	458	9.977 055	46
1.818	0.499 816	412	0.476 917	458	9.977 101	46
1.819	0.500 228	412	0.477 375	457	9.977 147	45
1.820	0.500 640	412	0.477 832	458	9.977 192	46
1.821	0.501 052	412	0.478 290	458	9.977 238	45
1.822	0.501 464	413	0.478 748	457	9.977 283	46
1.823	0.501 877	412	0.479 205	458	9.977 329	45
1.824	0.502 289	412	0.479 663	457	9.977 374	45
1.825	0.502 701	412	0.480 120	458	9.977 419	45
1.826	0.503 113	413	0.480 578	457	9.977 464	45
1.827	0.503 526	412	0.481 035	458	9.977 509	45
1.828	0.503 938	413	0.481 493	457	9.977 554	45
1.829	0.504 351	412	0.481 950	457	9.977 599	45
1.830	0.504 763	413	0.482 407	457	9.977 644	45
1.831	0.505 176	412	0.482 864	457	9.977 689	44
1.832	0.505 588	413	0.483 321	458	9.977 733	45
1.833	0.506 001	412	0.483 779	457	9.977 778	44
1.834	0.506 413	413	0.484 236	457	9.977 822	45
1.835	0.506 826	413	0.484 693	457	9.977 867	44
1.836	0.507 239	413	0.485 150	457	9.977 911	44
1.837	0.507 652	412	0.485 607	457	9.977 955	44
1.838	0.508 064	413	0.486 064	456	9.977 999	44
1.839	0.508 477	413	0.486 520	457	9.978 043	44
1.840	0.508 890	413	0.486 977	457	9.978 087	44
1.841	0.509 303	413	0.487 434	457	9.978 131	43
1.842	0.509 716	413	0.487 891	456	9.978 174	44
1.843	0.510 129	413	0.488 347	457	9.978 218	44
1.844	0.510 542	413	0.488 804	456	9.978 262	43
1.845	0.510 955	413	0.489 260	457	9.978 305	43
1.846	0.511 368	414	0.489 717	456	9.978 348	44
1.847	0.511 782	413	0.490 173	457	9.978 392	43
1.848	0.512 195	413	0.490 630	456	9.978 435	43
1.849	0.512 608	414	0.491 086	457	9.978 478	43

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.850	0.513 022	413	0.491 543	456	9.978 521	43
1.851	0.513 435	413	0.491 999	456	9.978 564	43
1.852	0.513 848	414	0.492 455	456	9.978 607	43
1.853	0.514 262	413	0.492 911	456	9.978 650	42
1.854	0.514 675	414	0.493 367	457	9.978 692	43
1.855	0.515 089	413	0.493 824	456	9.978 735	42
1.856	0.515 502	414	0.494 280	456	9.978 777	43
1.857	0.515 916	414	0.494 736	456	9.978 820	42
1.858	0.516 330	413	0.495 192	456	9.978 862	42
1.859	0.516 743	414	0.495 648	455	9.978 904	43
1.860	0.517 157	414	0.496 103	456	9.978 947	42
1.861	0.517 571	413	0.496 559	456	9.978 989	42
1.862	0.517 984	414	0.497 015	456	9.979 031	42
1.863	0.518 398	414	0.497 471	456	9.979 073	41
1.864	0.518 812	414	0.497 927	455	9.979 114	42
1.865	0.519 226	414	0.498 382	456	9.979 156	42
1.866	0.519 640	414	0.498 838	455	9.979 198	41
1.867	0.520 054	414	0.499 293	456	9.979 239	42
1.868	0.520 468	414	0.499 749	455	9.979 281	41
1.869	0.520 882	414	0.500 204	456	9.979 322	42
1.870	0.521 296	415	0.500 660	455	9.979 364	41
1.871	0.521 711	414	0.501 115	456	9.979 405	41
1.872	0.522 125	414	0.501 571	455	9.979 446	41
1.873	0.522 539	414	0.502 026	455	9.979 487	41
1.874	0.522 953	415	0.502 481	456	9.979 528	41
1.875	0.523 368	414	0.502 937	455	9.979 569	41
1.876	0.523 782	414	0.503 392	455	9.979 610	41
1.877	0.524 196	415	0.503 847	455	9.979 651	40
1.878	0.524 611	414	0.504 302	455	9.979 691	41
1.879	0.525 025	415	0.504 757	455	9.979 732	40
1.880	0.525 440	414	0.505 212	455	9.979 772	41
1.881	0.525 854	415	0.505 667	455	9.979 813	40
1.882	0.526 269	414	0.506 122	455	9.979 853	40
1.883	0.526 683	415	0.506 577	455	9.979 893	41
1.884	0.527 098	415	0.507 032	455	9.979 934	40
1.885	0.527 513	415	0.507 487	454	9.979 974	40
1.886	0.527 928	414	0.507 941	455	9.980 014	40
1.887	0.528 342	415	0.508 396	455	9.980 054	40
1.888	0.528 757	415	0.508 851	454	9.980 094	39
1.889	0.529 172	415	0.509 305	455	9.980 133	40
1.890	0.529 587	415	0.509 760	455	9.980 173	40
1.891	0.530 002	415	0.510 215	454	9.980 213	39
1.892	0.530 417	415	0.510 669	455	9.980 252	40
1.893	0.530 832	415	0.511 124	454	9.980 292	39
1.894	0.531 247	415	0.511 578	454	9.980 331	39
1.895	0.531 662	415	0.512 032	455	9.980 370	40
1.896	0.532 077	415	0.512 487	454	9.980 410	39
1.897	0.532 492	415	0.512 941	454	9.980 449	39
1.898	0.532 907	416	0.513 395	455	9.980 488	39
1.899	0.533 323	415	0.513 850	454	9.880 527	39

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Continued)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.900	0.533 738	415	0.514 304	454	9.980 566	39
1.901	0.534 153	416	0.514 758	454	9.980 605	38
1.902	0.534 569	415	0.515 212	454	9.980 643	39
1.903	0.534 984	415	0.515 666	454	9.980 682	39
1.904	0.535 399	416	0.516 120	454	9.980 721	38
1.905	0.535 815	415	0.516 574	454	9.980 759	39
1.906	0.536 230	416	0.517 028	454	9.980 798	38
1.907	0.536 646	415	0.517 482	454	9.980 836	38
1.908	0.537 061	416	0.517 936	454	9.980 874	39
1.909	0.537 477	416	0.518 390	453	9.980 913	38
1.910	0.537 893	415	0.518 843	454	9.980 951	38
1.911	0.538 308	416	0.519 297	454	9.980 989	38
1.912	0.538 724	416	0.519 751	454	9.981 027	38
1.913	0.539 140	416	0.520 205	453	9.981 065	38
1.914	0.539 556	415	0.520 658	454	9.981 103	37
1.915	0.539 971	416	0.521 112	453	9.981 140	38
1.916	0.540 387	416	0.521 565	454	9.981 178	38
1.917	0.540 803	416	0.522 019	453	9.981 216	37
1.918	0.541 219	416	0.522 472	454	9.981 253	38
1.919	0.541 635	416	0.522 926	453	9.981 291	37
1.920	0.542 051	416	0.523 379	453	9.981 328	37
1.921	0.542 467	416	0.523 832	454	9.981 365	38
1.922	0.542 883	416	0.524 286	453	9.981 403	37
1.923	0.543 299	416	0.524 739	453	9.981 440	37
1.924	0.543 715	417	0.525 192	453	9.981 477	37
1.925	0.544 132	416	0.525 645	454	9.981 514	37
1.926	0.544 548	416	0.526 099	453	9.981 551	37
1.927	0.544 964	416	0.526 552	453	9.981 588	37
1.928	0.545 380	417	0.527 005	453	9.981 624	37
1.929	0.545 797	416	0.527 458	453	9.981 661	37
1.930	0.546 213	416	0.527 911	453	9.981 698	36
1.931	0.546 629	417	0.528 364	453	9.981 734	37
1.932	0.547 046	416	0.528 817	453	9.981 771	36
1.933	0.547 462	417	0.529 270	453	9.981 807	37
1.934	0.547 879	416	0.529 723	452	9.981 844	36
1.935	0.548 295	417	0.530 175	453	9.981 880	36
1.936	0.548 712	416	0.530 628	453	9.981 916	36
1.937	0.549 128	417	0.531 081	453	9.981 952	36
1.938	0.549 545	417	0.531 534	452	9.981 988	36
1.939	0.549 962	416	0.531 986	453	9.982 024	36
1.940	0.550 378	417	0.532 439	452	9.982 060	36
1.941	0.550 795	417	0.532 891	453	9.982 096	36
1.942	0.551 212	417	0.533 344	453	9.982 132	36
1.943	0.551 629	417	0.533 797	452	9.982 168	35
1.944	0.552 046	417	0.534 249	452	9.982 203	36
1.945	0.552 463	416	0.0534 71	453	9.982 239	35
1.946	0.552 879	417	0.535 154	452	9.982 274	36
1.947	0.553 296	417	0.535 606	453	9.982 310	35
1.948	0.553 713	417	0.536 059	452	9.982 345	35
1.949	0.554 130	417	0.536 511	452	9.982 380	36

TABLES OF LOGARITHMS OF HYPERBOLIC FUNCTIONS
(Concluded)

x	log cosh x	d	log sinh x	d	log tanh x	d
1.950	0.554 547	417	0.536 963	452	9.982 416	35
1.951	0.554 964	418	0.537 415	453	9.982 451	35
1.952	0.555 382	417	0.537 868	452	9.982 486	35
1.953	0.555 799	417	0.538 320	452	9.982 521	35
1.954	0.556 216	417	0.538 772	452	9.982 556	35
1.955	0.556 633	417	0.539 224	452	9.982 591	35
1.956	0.557 050	418	0.539 676	452	9.982 626	34
1.957	0.557 468	417	0.540 128	452	9.982 660	35
1.958	0.557 885	417	0.540 580	452	9.982 695	34
1.959	0.558 302	418	0.541 032	452	9.982 729	35
1.960	0.558 720	417	0.541 484	452	9.982 764	34
1.961	0.559 137	418	0.541 936	451	9.982 798	35
1.962	0.559 555	417	0.542 387	452	9.982 833	34
1.963	0.559 972	418	0.542 839	452	9.982 867	34
1.964	0.560 390	417	0.543 291	452	9.982 901	35
1.965	0.560 807	418	0.543 743	451	9.982 936	34
1.966	0.561 225	417	0.544 194	452	9.982 970	34
1.967	0.561 642	418	0.544 646	452	9.983 004	34
1.968	0.562 060	418	0.545 098	451	9.983 038	34
1.969	0.562 478	417	0.545 549	452	9.983 072	33
1.970	0.562 895	418	0.546 001	451	9.983 105	34
1.971	0.563 313	418	0.546 452	452	9.983 139	34
1.972	0.563 731	418	0.546 904	451	9.983 173	33
1.973	0.564 149	417	0.547 355	451	9.983 206	34
1.974	0.564 566	418	0.547 806	452	9.983 240	34
1.975	0.564 984	418	0.548 258	451	9.983 274	33
1.976	0.565 402	418	0.548 709	451	9.983 307	33
1.977	0.565 820	418	0.549 160	452	9.983 340	34
1.978	0.566 238	418	0.549 612	451	9.983 374	33
1.979	0.566 656	418	0.550 063	451	9.983 407	33
1.980	0.567 074	418	0.550 514	451	9.983 440	33
1.981	0.567 492	418	0.550 965	451	9.983 473	33
1.982	0.567 910	418	0.551 416	452	9.983 506	33
1.983	0.568 328	419	0.551 868	451	9.983 539	33
1.984	0.568 747	418	0.552 319	451	9.983 572	33
1.985	0.569 165	418	0.552 770	451	9.983 605	33
1.986	0.569 583	418	0.553 221	451	9.983 638	32
1.987	0.570 001	419	0.553 672	450	9.983 670	33
1.988	0.570 420	418	0.554 122	451	9.983 703	32
1.989	0.570 838	418	0.554 573	451	9.983 735	33
1.990	0.571 256	419	0.555 024	451	9.983 768	32
1.991	0.571 675	418	0.555 475	451	9.983 800	33
1.992	0.572 093	418	0.555 926	451	9.983 833	32
1.993	0.572 511	419	0.556 377	450	9.983 865	32
1.994	0.572 930	418	0.556 827	451	9.983 897	33
1.995	0.573 348	419	0.557 278	451	9.983 930	32
1.996	0.573 767	419	0.557 729	450	9.983 962	32
1.997	0.574 186	418	0.558 179	451	9.983 994	32
1.998	0.574 604	419	0.558 630	450	9.984 026	32
1.999	0.575 023	418	0.559 080	451	9.984 058	31
2.000	0.575 441		0.559 531		9.984 089	

CURVE NO.

FORM A

SEPARATION OF ODD AND EVEN HARMONICS FOR 72 ORDINATE ANALYSIS

$y_m' = \frac{1}{2}(y_m - y_{m+36}).$ $y_m'' = \frac{1}{2}(y_m + y_{m+36}).$

m	Data		m	Diff.	Sum.	m	Data		m	Diff.	Sum.
	y_m	y_m		$2y_m'$	$2y_m''$		y_m	y_m		$2y_m'$	$2y_m''$
0			36			18			54		
1			37			19			55		
2			38			20			56		
3			39			21			57		
4			40			22			58		
5			41			23			59		
6			42			24			60		
7			43			25			61		
8			44			26			62		
9			45			27			63		
10			46			28			64		
11			47			29			65		
12			48			30			66		
13			49			31			67		
14			50			32			68		
15			51			33			69		
16			52			34			70		
17			53			35			71		

Use y_m' in analysis for odd harmonics. Carry y_m'' to Form E as data.

Remarks.

Computed by

FORM B

Analysis for Odd Harmonics from 36 Ordinates per Half Wave, or using y' from Form A as Data

Use in Form C

m	Data y_m	Data y_m	m	Diff. D_m	m	Sum S_m	
0					0		
1			35		1		$D_0' = D_0 - D_{12} =$
2			34		2		$D_1' = D_1 - D_{11} - D_{13} =$
3			33		3		$D_2' = D_2 - D_{10} - D_{14} =$
4			32		4		$D_3' = D_3 - D_9 - D_{15} =$
5			31		5		$D_4' = D_4 - D_8 - D_{16} =$
6			30		6		$D_5' = D_5 - D_7 - D_{17} =$
7			29		7		$D_0'' = D_0' - D_4' =$
8			28		8		$D_1'' = D_1' - D_3' - D_5' =$
9			27		9		
10			26		10		$S_1' = S_1 + S_{11} - S_{13} =$
11			25		11		$S_2' = S_2 + S_{10} - S_{14} =$
12			24		12		$S_3' = S_3 + S_9 - S_{15} =$
13			23		13		$S_4' = S_4 + S_8 - S_{16} =$
14			22		14		$S_5' = S_5 + S_7 - S_{17} =$
15			21		15		$S_6' = S_6 - S_{18} =$
16			20		16		$S_1'' = S_1' + S_3' - S_5' =$
17			19		17		$S_2'' = S_2' - S_6' =$
18					18		

$C_n^2 = A_n^2 + B_n^2$. $A_n \cos n\theta + B_n \sin n\theta = C_n \cos(n\theta - \phi_n)$. $\operatorname{Tan} \phi_n = \dfrac{B_n}{A_n}$.

Results

n	A_n	B_n	C_n	ϕ_n
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				
21				
22				
23				
24				
25				
26				
27				
28				
29				
30				
31				
32				
33				
34				
35				
36				

Computed by

Curve No.

FORM D

Check on Analysis for Odd Harmonics as per Forms B and C

Enter the same numbered A or B coefficient in blank space opposite numbers, with indicated algebraic sign

A, or cosine coefficients								B, or sine coefficients							
n	$\pm A_n$	n	$\pm A_n$	n	$\pm A_n$	n	$\pm A_n$	n	$\pm B_n$	n	$\pm B_n$	n	$\pm B_n$	n	$\pm B_n$
1		1						1							
− 3						3				3					
− 5				5				5				5		5	
7				7				− 7						− 7	
9						9				− 9		− 9			
− 11		11						− 11				− 11			
− 13		13						13				13			
15						15				15		15			
17				17				17						17	
− 19				19				− 19				− 19		− 19	
− 21						21				− 21					
23		23						− 23							
25		25						25							
− 27						27				27					
− 29				29				29				29		29	
31				31				− 31						− 31	
33						33				− 33		− 33			
− 35		35						− 35				− 35			
$G =$		$H =$		$J =$		$K =$		$A =$		$B =$		$E =$		$F =$	

$M = H - J =$ $\qquad$ $C = A + B - 2E =$ $\qquad$ $D = A - 2F =$

$y_0 = H + J + K$

$y_6 = 0.5\,A + B + 0.866\,M$

$y_{30} = 0.5\,A + B - 0.866\,M$

$y_{18} = A - B$

$y_{12} = 0.866\,D + 0.5\,y_0 - 1.5\,K$

$y_{24} = 0.866\,D - 0.5\,y_0 + 1.5\,K$

$y_9 = 0.707\,(C + G)$

$y_{27} = 0.707\,(C - G)$

Comparison

m	Ordinate computed	Ordinate from data
0		
6		
9		
12		
18		
24		
27		
30		

Curve No.

Computed by

CURVE NO.

FORM E

SEPARATION OF ODD AND EVEN HARMONICS FOR 36 ORDINATE ANALYSIS

$y_m' = \frac{1}{2}(y_m - y_{m+18})$. $y_m'' = \frac{1}{2}(y_m + y_{m+18})$.

m	Data		m	Diff.	Sum.
	y_m	y_m		$2\,y_m'$	$2\,y_m''$
0			18		
1			19		
2			20		
3			21		
4			22		
5			23		
6			24		
7			25		
8			26		
9			27		
10			28		
11			29		
12			30		
13			31		
14			32		
15			33		
16			34		
17			35		

Use y_m' in analysis for odd harmonics. Carry y_m'' to Form H as data.

Remarks.

Computed by

FORM F

Analysis for Odd Harmonics only using 18 Ordinates per Half Wave or y' from Form E as Data

Use in Form G

m	Data y_m	Data y_m	m	Diff. D_m	m	Sum S_m	
0		—			0	—	
1			17		1		$D_0' = D_0 - D_6 =$
2			16		2		$D_1' = D_1 - D_5 - D_7 =$
3			15		3		$D_2' = D_2 - D_4 - D_8 =$
4			14		4		$D_0'' = D_0' - D_2' =$
5			13		5		
6			12		6		$S_1' = S_1 + S_5 - S_7 =$
7			11		7		$S_2' = S_2 + S_4 - S_8 =$
8			10		8		$S_3' = S_3 - S_9 =$
9		—		—	9		$S_1'' = S_1' - S_3' =$

$C_n^2 = A_n^2 + B_n^2$. $\quad \operatorname{Tan} \phi_n = \dfrac{B_n}{A_n}$. $\quad A_n \cos n\theta + B_n \sin n\theta = C_n \cos (n\theta - \phi_n)$.

Results

n	A_n	B_n	C_n	ϕ_n
0		—		—
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18		—	—	—

Check on Analysis for Odd Harmonics as per Form G

Enter the same numbered A or B coefficient in blank space opposite numbers, with indicated algebraic sign

A, or cosine coefficients						B, or sine coefficients					
n	$\pm A_n$	n	$\pm A_n$	n	$\pm A_n$	n	$\pm B_n$	n	$\pm B_n$	n	$\pm B_n$
1						1					
				3				3			
		5				5				5	
		7				− 7				− 7	
				9				− 9			
11						−11					
13						13					
				15				15			
		17				17				17	
$H =$		$J =$		$K =$		$A =$		$B =$		$F =$	

$D = A - 2F =$
$M = H - J =$
$y_0 = H + J + K =$

$y_3 = 0.5\,A + B + 0.866\,M.$
$y_{15} = 0.5\,A + B - 0.866\,M.$
$y_9 = A - B.$

$y_6 = 0.866\,D + 0.5\,y_0 - 1.5\,K.$
$y_{12} = 0.866\,D - 0.5\,y_0 + 1.5\,K.$

Comparison

m	Ord. comp.	Ord. data
0		
3		
6		
9		
12		
15		

Curve No.

Computed by

FORM G

Analysis for Odd Harmonics only, from Form F

$\theta°$	$\cos\theta$	Enter same numbered diff. (D_m) from Form F multiplied by $\cos\theta$, with indicated algebraic sign																	
0	1.00000	0				0′				0				0				0″	
10	0.98481				1								7				5		
20	0.93969	2								−4				−8					
30	0.86603				3				1′				−3				−3		
40	0.76604	4								8				−2					
50	0.64279				5								1				−7		
60	0.50000	6				2′				6				6					
70	0.34202				7								−5				1		
80	0.17365	8								−2				4					
Sums I, II		I			II	I			II	I			II	I			II	I	
I + II																			
I − II																			
$\frac{1}{9}$(I + II)		A_1 =				A_3 =				A_5 =				A_7 =				A_9 =	
$\frac{1}{9}$(I − II)		A_{17} =				A_{15} =				A_{13} =				A_{11} =					

$\theta°$	$\sin\theta$	Enter same numbered sums (S_m) from Form F multiplied by $\sin\theta$, with indicated algebraic sign																	
10	0.17365	1								−7				−5					
20	0.34202				2								−4				−8		
30	0.50000	3				1′				3				−3					
40	0.64279				4								8				2		
50	0.76604	5								1				7					
60	0.86603				6				2′				−6				6		
70	0.93969	7								−5				1					
80	0.98481				8								2				−4		
90	1.00000	9				3′				9				−9				1″	
Sums I, II		I			II	I			II	I			II	I			II	I	
I + II																			
I − II																			
$\frac{1}{9}$(I + II)		B_1 =				B_3 =				B_5 =				B_7 =				B_9 =	
$\frac{1}{9}$(I − II)		B_{17} =				B_{15} =				B_{13} =				B_{11} =					

Curve No.

Computed by

FORM H

Analysis for Odd and Even Harmonics using 18 Ordinates per Wave or y'' from Form E as Data

Use in Form I

m	Data y_m	Data y_m	m	Sum. S_m	m	Diff. D_m	
0		—			0	—	$S_0' = S_0 + S_6 =$
1			17		1		$S_1' = S_1 + S_5 + S_7 =$
2			16		2		$S_2' = S_2 + S_4 + S_8 =$
3			15		3		$S_3' = S_3 + S_9 =$
4			14		4		$S_0'' = S_0' + S_2' =$
5			13		5		$S_1'' = S_1' + S_3' =$
6			12		6		
7			11		7		$D_1' = D_1 - D_5 + D_7 =$
8			10		8		$D_2' = D_2 - D_4 + D_8 =$
9		—			9	—	

Results

n	A_n	B_n	C_n	ϕ_n
0		—		—
1				
2				
3				
4				
5				
6				
7				
8				
9		—	—	—

$$C_n^2 = A_n^2 + B_n^2. \quad \operatorname{Tan} \phi_n = \frac{B_n}{A_n}. \qquad A_n \cos n\theta + B_n \sin n\theta = C_n \cos (n\theta - \phi_n).$$

Check on Analysis for Even and Odd Harmonics from Form I

$H_0 = A_0 + A_6 =$

$H_1 = A_1 + A_5 + A_7 =$

$H_2 = A_2 + A_4 + A_8 =$

$H_3 = A_3 + A_9 =$

$G_1 = B_1 - B_5 + B_7 =$

$G_2 = B_2 - B_4 + B_8 =$

$y_0 = H_0 + H_1 + H_2 + H_3$

$y_9 = H_0 - H_1 + H_2 - H_3$

$y_3 = H_0 - H_3 + 0.5\,(H_1 - H_2) + 0.866\,(G_1 + G_2)$

$y_{15} = H_0 - H_3 + 0.5\,(H_1 - H_2) - 0.866\,(G_1 + G_2)$

$y_6 = H_0 + H_3 - 0.5\,(H_1 + H_2) + 0.866\,(G_1 - G_2)$

$y_{12} = H_0 + H_3 - 0.5\,(H_1 + H_2) - 0.866\,(G_1 - G_2)$

Comparison

Curve No.

m	Ord. comp.	Ord. data
0		
3		
6		
9		
12		
15		

Computed by

FORM I

Analysis for Odd and Even Harmonics from Form H

Cosine

$\theta°$	$\cos\theta$	Enter same numbered sums (S_m) from Form H multiplied by $\cos\theta$, with indicated algebraic sign																			
0	1.00000	0			−9	0			9	0′			−3′	0			9	0″			1″
20	0.93969	−8			1	−4			−5					−2			−7				
40	0.76604	2			−7	8			1					4			5				
60	0.50000	−6			3	−6			−3	− 2′			1′	−6			−3				
80	0.17365	4			−5	2			7					8			1				
Sums I, II		I			II	I			II	I			II	I			II	I			II
I + II																					
I − II																					
$\frac{1}{9}$ (I + II)		$A_1 =$				$A_2 =$				$A_3 =$				$A_4 =$				$2A_0 =$			
$\frac{1}{9}$ (I − II)		$A_8 =$				$A_7 =$				$A_6 =$				$A_5 =$				$2A_9 =$			

Sine

$\theta°$	$\sin\theta$	Enter same numbered diff. (D_m) from Form H multiplied by $\sin\theta$ with indicated algebraic sign															
20	0.34202	1			8	− 5			4					− 7			2
40	0.64279	7			2	1			− 8					5			− 4
60	0.86603	3			6	3			− 6	1′			2′	− 3			6
80	0.98481	5			4	− 7			2					1			− 8
Sums I, II		I			II	I			II	I			II	I			II
I + II																	
I − II																	
$\frac{1}{9}$ (I + II)		$B_1 =$				$B_2 =$				$B_3 =$				$B_4 =$			
$\frac{1}{9}$ (I − II)		$B_8 =$				$B_7 =$				$B_6 =$				$B_5 =$			

Curve No.

Computed by